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Ultraspherical Integration Method for Solving Beam Bending Boundary Value Problem

M. El-Kady

Mathematics Department, Faculty of Science, Helwan University,Egypt, mam_el_kady@yahoo.com

M.A.Attia*

Mathematics Department, College Of Science Al-Zulfi, Majmaah University, Saudi Arabia, ma.attia@mu.edu.sa

F. Ragab

Mathematics Department, Faculty of Science, Aswan University, Egypt, umix_paints2020@yahoo.com

Abstract

In this paper, we applied successive ultraspherical integration matrices, which are used for the numerical solution of fourth order linear boundary value problem arising in bending of a rectangular beam on elastic foundation. This method is used to approximate for the highest-order derivative and generating approximations to the lower-order derivatives through integration of the highest-order derivative. We can then use the produced equations in the form of algebraic system and hence it is converted to nonlinear programming. Numerical examples illustrated the accuracy and efficiency of the proposed method

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1. Introduction

Fourth order linear BVP has been solved by various numerical methods. Shahid S. Siddiqi et al. [9] have used non polynomial spline to solve fourth-order boundary value problems. Siraj-ul-Islam et al. developed a technique based on quartic non-polynomial spline functions for approximations to the solution of a system of fourth-order. Riaz A. Usmani developed the method of the solution for fourth-order boundary value problem, considering it to be the problem of bending a rectangular clamped beam of length resting on an elastic foundation.

The aim of this paper is to present ultraspherical spectral integration matrices depends on using ultraspherical polynomials for solving beam bending boundary value problem.

The organization of this paper is as follows: In section 2, we introduce ultraspherical polynomials and some of its properties. In section 3, we presented the procedure of the steps of ultraspherical spectral integration method. In section 4, we introduce a Description of the used method. In section 5, we present numerical results demonstrating the accuracy of our methods for some example of beam bending boundary value problem. Section 6, contains the conclusion of this paper.

2. Ultraspherical Polynomials and Some Properties:

The ultraspherical or Gegenbauer polynomials with the real parameter ($\alpha > -1/2$ $,\alpha \neq 0$), are a sequence of polynomials $C_j^{(\alpha)}(x)$, $j = 0,1,2,\dots$. In the finite domain $x \in [-1,1]$, each

* Corresponding author

degree j satisfies the orthogonality relation respectively as follows:

$$\int_{-1}^{1}(1-x^2)^{\alpha-\frac{1}{2}}C_j^{(\alpha)}(x)C_k^{(\alpha)}(x)dx=\begin{cases}0, & j\neq k,\\ \psi_j^{(\alpha)}, & j=k,\end{cases} \quad (2.1)$$

Here the normalization constant $\psi_j^{(\alpha)}$ is defined as Szegö:

$$\psi_j^{(\alpha)}=2^{1-2\alpha}\pi\frac{\Gamma(j+2\alpha)}{(j+\alpha)\{\Gamma(\alpha)\}^2\Gamma(j+1)},\ \alpha\neq 0 \quad (2.2)$$

The polynomials may be generated by the Rodrigue's formula given as Bell:

$$C_j^{(\alpha)}(x)=\sum_{r=0}^{[j/2]}(-1)^r\frac{\Gamma(j-r+\alpha)}{\Gamma(\alpha)(r!)(j-2r)!}(2x)^{j-2r} \quad (2.3)$$

where $[j/2]$ refers to the integer part of the fraction.

The general expressions for ultraspherical polynomials can be put in the following way:

$$C_j^{(\alpha)}(x)=\sum_{r=0}^{[j/2]}G_r^j(\alpha)\,x^{j-2r}, \quad (2.4)$$

where

$$G_r^j(\alpha)=(-1)^r\frac{2^{j-2r}\Gamma(j-r+\alpha)}{\Gamma(\alpha)(r!)(j-2r)!}. \quad (2.5)$$

A relation between the coefficients $G_{r+1}^j(\alpha)$ and $G_r^j(\alpha)$ is given by

$$G_{r+1}^j(\alpha)=-\frac{(j-2r-1)(j-2r-2)}{4(r+1)(\alpha+j-r-1)}G_r^j(\alpha) \quad (2.6)$$

In particular, we have the special values

$$G_0^0(\alpha)=1,\qquad G_0^j(\alpha)=2^j\frac{\Gamma(j+\alpha)}{\Gamma(\alpha)\Gamma(j+1)}.$$

The fundamental recurrent formulae for ultraspherical polynomials are defined as

$$\begin{aligned}(j+1)C_{j+1}^{(\alpha)}(x)&=2(\alpha+j)xC_j^{(\alpha)}(x)\\&-(2\alpha+j-1)C_{j-1}^{(\alpha)}(x)\end{aligned}, \quad (2.7)$$

with the first two being: $C_0^{(\lambda)}(x)=1$, $C_1^{(\lambda)}(x)=2\lambda x$.

Theorem (2.1):

The m-th integral of the ultraspherical polynomials (2.4) is expressed in terms of ultraspherical polynomials as follows:

$$\begin{aligned}(I_mC_j^{(\alpha)})(x)&=\int_{-1}^{x}\int_{-1}^{t_{m-1}}\dots\int_{-1}^{t_1}C_j^{(\alpha)}(t_0)\,dt_0\,dt_1\dots dt_{m-1}\\&=\sum_{r=0}^{[j/2]}\gamma_{r,j}^{(m)}G_r^j(\alpha)x^{j-2r+m}\\&+\sum_{l=1}^{m}(-1)^{m-l}\frac{(x+1)^{l-1}}{(l-1)!}\mathrm{E}_j^{(m-l+1)}(\alpha)\end{aligned}, \quad (2.8)$$

where

$$\gamma_{r,j}^{(m)}=\prod_{l=1}^{m}\frac{1}{(j-2r+l)};$$

$$\mathrm{E}_j^{(m)}(\alpha)=\sum_{r=0}^{[j/2]}(-1)^{j-2r+2}\gamma_{r,j}^{(m)}G_r^j(\alpha),\ m\geq 1.$$

Proof: see M. A. Ibrahim

We used an approximation of any continuous function $f(x)$ and an approximation of their integrals by interpolating the function with ultraspherical polynomials at two sets of nodes:

- The set of equally spaced points:

$$S_1=\{x_i=-1+\frac{2i}{N}, i=0,1,..,N\}.$$

- The set of zeros points of the ultraspherical polynomials:

$$S_2=\left\{x_i : C_{N+1}^{(\alpha)}(x_i)=0, i=0,1,..,N\right\}.$$

Theorem (2.2):

If $f_N(x)$ is the ultraspherical approximation to a function $f(x)$ in finite expansion, i.e.

$$f_N(x)=\sum_{j=0}^{N}a_j\,C_j^{(\alpha)}(x), \quad (2.9)$$

then

$$a_j = \left(\psi_j^{(\alpha)}\right)^{-1} \int_{-1}^{1} (1-x^2)^{\alpha-\frac{1}{2}}\ C_j^{(\alpha)}(x)\, f(x)\, dx \quad (2.10)$$

Proof: See El-Hawary et al.

3. Ultraspherical Integration Matrices

Many authors presented spectral integration matrices proven successful in the numerical approximation of many types of differential equations such as. Elbarbary presented spectral successive integration matrix where it can be used to construct a Chebyshev expansion method for the solution of boundary value problems. We approximate the integral of a function $f(x)$ by interpolating the function with ultraspherical polynomials at the points S_1 and S_2.

Theorem (3.2):

If $f(x)$ is approximated by ultraspherical polynomials, then the *m*-th integral of $f(t)$ is approximated by ultraspherical expansion in the form:

$$\int_{-1}^{x_i}\int_{-1}^{t_{m-1}} \cdots \int_{-1}^{t_1} f(t_0)\, dt_0\, dt_1 \ldots dt_{m-1} = \sum_{k=0}^{N} q_{i,k}^{(m)}(\alpha)\, f(x_k) \quad (2.11)$$

where the entries of m-th ultraspherical integration matrices $q_{i,k}^{(m)}(\alpha)$, $i,k = 0,1,\ldots,N$ are given as follows:

Case 1: $x \in S_1$

$$q_{i,k}^{(m)}(\alpha) = \sum_{j=0}^{N}\sum_{r=0}^{[j/2]} \frac{2\theta_k\, \gamma_{r,j}^{(m)}\, G_r^j(\alpha)}{N} \left(\psi_j^{(\alpha)}\right)^{-1} (1-x_k^2)^{\alpha-\frac{1}{2}}\, C_j^{(\alpha)}(x_k)\, x_i^{j-2r+m} + \sum_{j=0}^{N}\sum_{l=1}^{m} \frac{2\theta_k (-1)^{m-l}(x_i+1)^{l-1}}{N(l-1)!} \left(\psi_j^{(\alpha)}\right)^{-1} (1-x_k^2)^{\alpha-\frac{1}{2}}\ C_j^{(\alpha)}(x_k)\, \mathrm{E}_j^{(m-l+1)}(\alpha)$$

Case 2: $x \in S_2$

$$q_{i,k}^{(m)}(\alpha) = \sum_{j=0}^{N}\sum_{r=0}^{[j/2]} \gamma_{r,j}^{(m)}\, G_r^j(\alpha) \left(\psi_j^{(\alpha)}\right)^{-1} \varpi_k^{(\alpha)}\, C_j^{(\alpha)}(x_k)\, x_i^{j-2r+m} + \sum_{j=0}^{N}\sum_{l=1}^{m} \frac{(-1)^{m-l}(x_i+1)^{l-1}}{(l-1)!} \mathrm{E}_j^{(m-l+1)}(\alpha) \left(\psi_j^{(\alpha)}\right)^{-1} \varpi_k^{(\alpha)}\, C_j^{(\alpha)}(x_k)$$

Proof: See M. A. Ibrahim

4. Description of the method

We consider a general fourth order boundary value problem given by

$$y^{(4)}(x) + f(x)h(y) = g(x),\ \ x \in [a,b], \quad (4.1)$$

Subject to the boundary conditions:

$$y(a) = \nu_0, \qquad y(b) = \beta_0,$$
$$y^{(1)}(a) = \nu_1, \qquad y^{(1)}(b) = \beta_1.$$

Where ν_0, ν_1, β_0 and β_1 are finite real constants, and $f(x), h(y)$ and $g(x)$ are continuous functions on the interval $[a,b]$.

By applying ultraspherical integration method (2.11), the highest derivative of y(x) can be written as

$$y^{(4)}(x) = \Phi(x). \quad (4.2)$$

The low-order derivatives $\partial^k y(x_i)/\partial x^k$, $k = 0,1,2,\ \ i = 0,1,\ldots,N$ are generated through integration of equation as follows

$$y^{(3)}(x_i) = \int_a^{x_i} \Phi(x)\, dx + c_1 \quad (4.3)$$

$$y^{(2)}(x_i) = \int_a^{x_i}\int_a^{x_i} \Phi(x) dx dx + (x_i - a)c_1 + c_2 \quad (4.4)$$

$$y^{(1)}(x_i) = \int_a^{x_i}\int_a^{x_i}\int_a^{x_i} \Phi(x) dx dx dx + \frac{1}{2}(x_i - a)^2 c_1 + (x_i - a)c_2 + c_3 \quad (4.5)$$

$$y(x_i) = \int_a^{x_i}\int_a^{x_i}\int_a^{x_i}\int_a^{x_i} \Phi(x)dxdxdxdx + \frac{1}{6}(x_i - a)^3 c_1 + \frac{1}{2}(x_i - a)^2 c_2 + (x_i - a)c_3 + c_4 \quad (4.6)$$

The successive integration of equations (4.2) to (4.6) is approximated by ultraspherical integration method as follows:

$$y^{(3)}(x_i) = \sum_{j=0}^{N} q_{i,j}^{(1)}(\alpha)\Phi(x_j) + c_1,$$

$$y^{(2)}(x_i) = \sum_{j=0}^{N} q_{i,j}^{(2)}(\alpha)\Phi(x_j) + (x_i - a)c_1 + c_2,$$

$$y^{(1)}(x_i) = \sum_{j=0}^{N} q_{i,j}^{(3)}(\alpha)\Phi(x_j) + \frac{1}{2}(x_i - a)^2 c_1 + (x_i - a)c_2 + c_3$$

$$y(x_i) = \sum_{j=0}^{N} q_{i,j}^{(4)}(\alpha)\Phi(x_j) + \frac{1}{6}(x_i - a)^3 c_1 + \frac{1}{2}(x_i - a)^2 c_2 + (x_i - a)c_3 + c_4.$$

Then we can determine the approximation solution $y(x_i)$ by determining the coefficients c_i, $i = 1, 2, 3, 4$ from the boundary conditions of the problem of fourth order boundary value problem.

By Substituting the approximation solution in equation of the problem of fourth boundary value problem (4.1) and then we obtain the unconstrained optimization problem, which can be written as

Minimize

$$\mathbf{F} = \mathbf{F}(\Phi, \alpha) \quad (4.7)$$

Where

$$\Phi = \left[\Phi_0(x), \Phi_1(x), \ldots, \Phi_N(x)\right].$$

The unconstrained optimization problem (4.7) can be solved using partial quadratic interpolation method (El-Gindy).

5. Numerical examples

In this section, we will use ultraspherical integration method to get an approximate solution in solving the beam bending boundary value problems, the clamped-clamped beam which belongs to the general class of the boundary value problems in the form:

$$y^{(4)}(x) + f(x)p(y) = g(x), \quad x \in [0,1],$$
$$y(0) = y(1) = y^{(1)}(0) = y^{(1)}(1) = 0.$$

Example 5.1

Consider the following boundary value problem which describes the model of the bending of a thin beam clamped at both ends:

$$y^{(4)} = (x^4 + 14x^3 + 49x^2 + 32x - 12)e^x, \quad x \in [0,1] \quad (5.1)$$

Subject to the boundary condition

$$y(0) = y(1) = 0, \quad y^{(1)}(0) = y^{(1)}(1) = 0. \quad (5.2)$$

The analytic solution of the above system solution is:

$$y(x) = x^2(1-x)^2 e^x$$

We apply ultraspherical integration method (2.11) the highest derivative of y can be written as

$$y^{(4)}(x) = \Phi(x). \quad (5.3)$$

The low-order derivatives $\partial^k y(x_i)/\partial x^k$, $k = 0,1,2,3$; $i = 0,1,\ldots,N$ are generated through integration of equation (5.3) as follows

$$y^{(3)}(x) = \int_0^x \Phi(x)dx + c_1 \quad (5.4)$$

$$y^{(2)}(x) = \int_0^x\int_0^x \Phi(x)dxdx + xc_1 + c_2 \quad (5.5)$$

$$y^{(1)}(x) = \int_0^x\int_0^x\int_0^x \Phi(x)dxdxdx + \frac{1}{2}x^2 c_1 + xc_2 + c_3 \quad (5.6)$$

$$y(x) = \int_0^x\int_0^x\int_0^x\int_0^x \Phi(x)dxdxdxdx + \frac{1}{6}x^3 c_1 + \frac{1}{2}x^2 c_2 + xc_3 + c_4 \quad (5.7)$$

The constants $c_i, i = 1,2,3,4$ can be determined from the boundary conditions these constants are found to be

$$c_4 = 0 \quad (5.8)$$

$$c_3 = 0 \quad (5.9)$$

$$c_1 = 12\sum_{k=0}^{N} q_{N,k}^{(4)}(\alpha)\Phi(x_k) - 6\sum_{k=0}^{N} q_{N,k}^{(3)}(\alpha)\Phi(x_k) \quad (5.10)$$

$$c_2 = -6\sum_{k=0}^{N} q_{N,k}^{(4)}(\alpha)\Phi(x_k) + 2\sum_{k=0}^{N} q_{N,k}^{(3)}(\alpha)\Phi(x_k) \quad (5.11)$$

Substituting from equations (5.8) to (5.11) in equation (5.7)

$$y(x_i) = \sum_{k=0}^{N} q_{i,k}^{(4)}(\alpha)\Phi(x_k) + \frac{1}{6}x_i^3(12\sum_{k=0}^{N} q_{N,k}^{(4)}(\alpha)\Phi(x_k) - 6\sum_{k=0}^{N} q_{N,k}^{(3)}(\alpha)\Phi(x_k)) + \frac{1}{2}x_i^2(-6\sum_{k=0}^{N} q_{N,k}^{(4)}(\alpha)\Phi(x_k) + 2\sum_{k=0}^{N} q_{N,k}^{(3)}(\alpha)\Phi(x_k)) \quad (5.12)$$

Then by substituting $y(x_i)$ in the equation (5.1) and we can be written as:

Minimize

$$\mathbf{F} = \mathbf{F}(\Phi, \alpha), \quad (5.13)$$

Where

$$\Phi = [\Phi_0(t), \Phi_1(t), ..., \Phi_N(t)].$$

It can be solved by using partial quadratic interpolation method (El-Gindy).

The following table presents the maximum absolute error, obtained by using ultraspherical integration methods at the points given in S_1, S_2 and Fig 5.1 mad a comparative between the approximate solution and the exact solution for N=10 In S_2.

Table (5.1): The maximum absolute error by ultraspherical integration Methods at $x \in S_1$, $x \in S_2$.

N	$x \in S_1$		$x \in S_2$	
	α	***MAE***	α	***MAE***
4	-0.22	3.94E-02	-0.49	1.93E-02
6	0.49	8.24E-04	-0.25	2.80E-04
8	-0.10	6.03E-06	0. 49	1.18E-06
10	0.75	2.17E-08	0.12	2.57E-09
12	0.49	2.09E-09	0.22	2.13E-09
16	0.499	2.13E-09	-0.49	2.11E-09

Table (5.2): Comparison between exact solution with the result in approximation solution to Ultraspherical integration method for N=10 in S_1.

x	**exact solution**	**Ultraspherical integration method**	**error**
0.1	0.00895188	0.00895191	2.17E-08
0.2	0.03126791	0.03126793	1.78E-08
0.3	0.05952877	0.05952879	1.81E-08
0.4	0.08592910	0.08592912	1.70E-08
0.5	0.10304508	0.10304510	1.64E-08
0.6	0.10495404	0.10495406	1.53E-08
0.7	0.08880649	0.08880651	1.46E-08
0.8	0.05697385	0.05697386	1.30E-08
0.9	0.01992279	0.01992280	1.52E-08

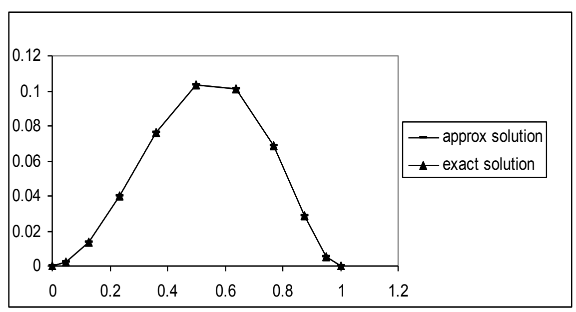

Fig 5.1: the approximate solution and the exact solution for N=10 In S_2.

Example 5.2

Consider the following nonlinear fourth-order B.V.Ps:

$$y^{(4)} + \frac{x^2}{1+y^2} = -72\left(1-5x+5x^2\right) + \frac{x^2}{1+(x-x^2)^6}, \quad (5.14)$$

$x \in [0,1]$

Subject to the boundary conditions:

$$y(0)=0, \qquad y(1)=0,$$
$$y^{(1)}(0)=0, \quad y^{(1)}(1)=0.$$

This has the analytic solution given by:

$$y(x) = x^3(1-x)^3.$$

We apply ultraspherical integration method (2.11), the highest derivative of y can be written as

$$y^{(4)}(x) = \Phi(x). \qquad (5.15)$$

The low-order derivatives $\partial^k y(x_i)/\partial x^k$, $k=0,1,2,\ i=0,1,\ldots,N$ are generated through integration of equation (5.15) as follows

$$y^{(3)}(x) = \int_0^x \Phi(x)dx + c_1 \qquad (5.16)$$

$$y^{(2)}(x) = \int_0^x\int_0^x \Phi(x)dxdx + xc_1 + c_2 \qquad (5.17)$$

$$y^{(1)}(x) = \int_0^x\int_0^x\int_0^x \Phi(x)dxdxdx + \frac{1}{2}x^2c_1 + xc_2 + c_3 \qquad (5.18)$$

$$y(x) = \int_0^x\int_0^x\int_0^x\int_0^x \Phi(x)dxdxdxdx + \frac{1}{6}x^3c_1 + \frac{1}{2}x^2c_2 + xc_3 + c_4 \qquad (5.19)$$

The constants $c_i, i=1,2,3,4$ can be determined from the boundary conditions these constants are found to be

$$c_4 = 0 \qquad (5.20)$$

$$c_3 = 0 \qquad (5.21)$$

$$c_1 = 12\sum_{k=0}^{N} q_{N,k}^{(4)}(\alpha)\Phi(x_k) - 6\sum_{k=0}^{N} q_{N,k}^{(3)}(\alpha)\Phi(x_k) \qquad (5.22)$$

$$c_2 = 2\sum_{k=0}^{N} q_{N,k}^{(3)}(\alpha)\Phi(x_k) - 6\sum_{k=0}^{N} q_{N,k}^{(4)}(\alpha)\Phi(x_k) \qquad (5.23)$$

Substituting from equations (5.20) to (5.23) in equation (5.19), we get

$$\begin{aligned} y(x_i) &= \sum_{k=0}^{N} q_{i,k}^{(4)}(\alpha)\Phi(x_k) \\ &+ \frac{1}{6}x_i^3\Big(12\sum_{k=0}^{N} q_{N,k}^{(4)}(\alpha)\Phi(x_k) - 6\sum_{k=0}^{N} q_{N,k}^{(3)}(\alpha)\Phi(x_k)\Big) \\ &+ \frac{1}{2}x_i^2\Big(2\sum_{k=0}^{N} q_{N,k}^{(3)}(\alpha)\Phi(x_k) - 6\sum_{k=0}^{N} q_{N,k}^{(4)}(\alpha)\Phi(x_k)\Big) \end{aligned} \qquad (5.24)$$

Then by substituting $y(x_i)$ in the equation (5.14)) and we can be written as:

Minimize

$$\mathbf{F} = \mathbf{F}(\Phi, \alpha), \qquad (5.25)$$

Where

$$\Phi = \left[\Phi_0(t), \Phi_1(t), \ldots, \Phi_N(t)\right].$$

It solved by using partial quadratic interpolation method (El-Gindy).

The following table presents the maximum absolute error, obtained by using ultraspherical integration method at the points given in S_1, S_2 and Fig (5.2) mad a comparative between the approximate solution and the exact solution for N=10 In S_2.

Table (5.3): The maximum absolute error by ultraspherical integration methods at $x \in S_1$, $x \in S_2$.

N	$x \in S_1$		$x \in S_2$	
	α	***MAE***	α	***MAE***
4	0.13	3.30E-02	-0.33	1.66E-02
6	-0.11	1.36E-08	-0.37	1.31E-08
8	-0.35	1.35E-08	-0.11	1.21E-08
10	1.30	1.34E-08	0.49	1.35E-08
12	0.59	1.36E-08	-0.21	1.22E-08
16	0.49	1.35E-08	-0.22	1.35E-08

Table (5.4): Comparison between exact solution with the result in approximation solution to Ultraspherical integration method for N=10 in S_1 .

x	**exact solution**	**Ultraspherical integration method**	**error**
0.1	0.00072900	0.00072900	8.26E-10
0.2	0.00409600	0.00409600	2.98E-09
0.3	0.00926100	0.00926101	5.91E-09
0.4	0.01382400	0.01382401	9.00E-09
0.5	0.01562500	0.01562501	1.16E-08
0.6	0.01382400	0.01382401	1.33E-08
0.7	0.00926100	0.00926101	1.34E-08
0.8	0.00409600	0.00409601	1.16E-08
0.9	0.00072900	0.00072901	7.26E-09

6. Conclusion

In this study, ultraspherical integration method has been applied to obtain the numerical solutions for solving beam bending boundary value problem, at the set of equally spaced points or the set of zeros points. The numerical results demonstrated the efficiency and accuracy of the proposed scheme of the method. The numerical results obtained by the proposed method are in a good agreement with the exact solutions available in the literature. By considering that the accuracy of our method depends on specified value of the parameter ultraspherical α .

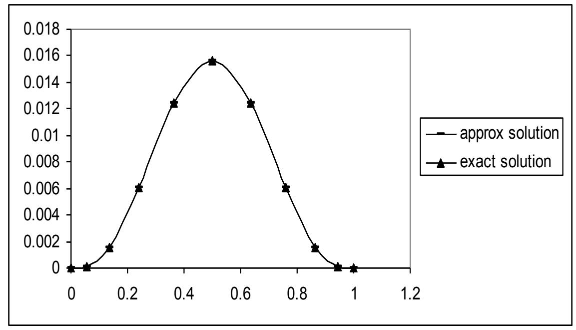

Fig 5.2: the approximate solution and the exact solution for N=10 In S_2.

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