Kingdom of Saudi Arabia
Ministry of Higher Education
Majmaa University
Faculty of Science

جامعة المجمعة
Majmaah University

Discrete Mathematics

Graph Theory
Basic Concepts & Applications

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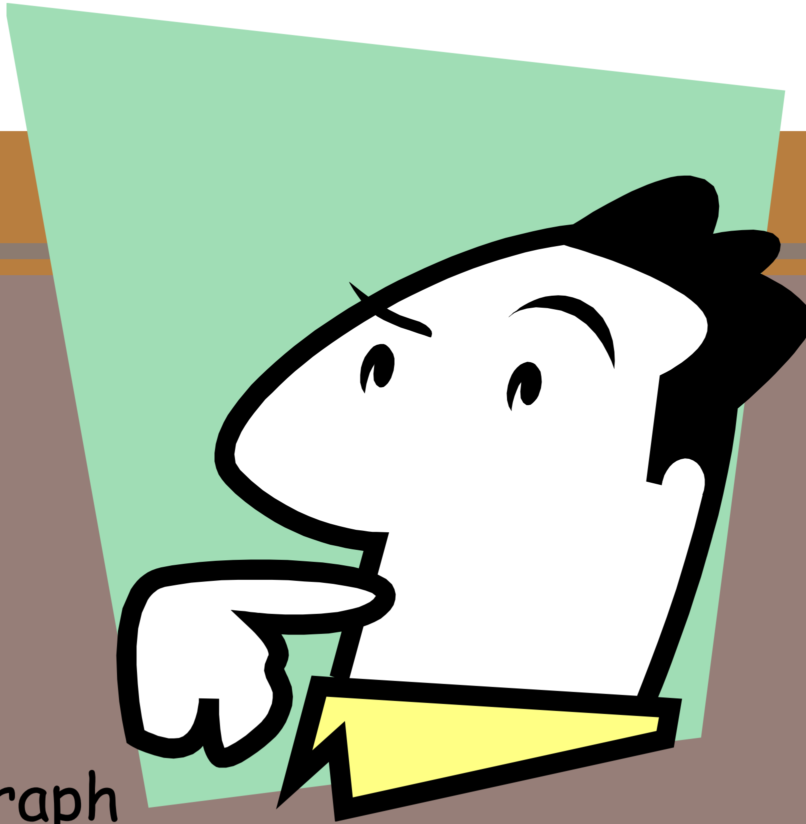
هجرية 1436 - 1435

Contents

1. Introduction To Graphs



- What is a Graph?
- Recognize different types of graph
- Understand the structure of a graph
- Show how graph are used as a model in variety of areas

What is a Graph ?

- A graph consists of A graph consists of two finite sets (sets having finitely many elements) :

     Set "V" of points called "*vertices*" (singular is vertex)

     Set "E" of connecting lines/arcs/line segments called "*edges*" .

Such that:

each edge connects two vertices (start and end at vertices), and these two vertices are called the "*endpoints*" of the edge. An edge that starts and ends at the same vertex is called a loop.

<u>Mathematically</u> : $G = G(V, E)$

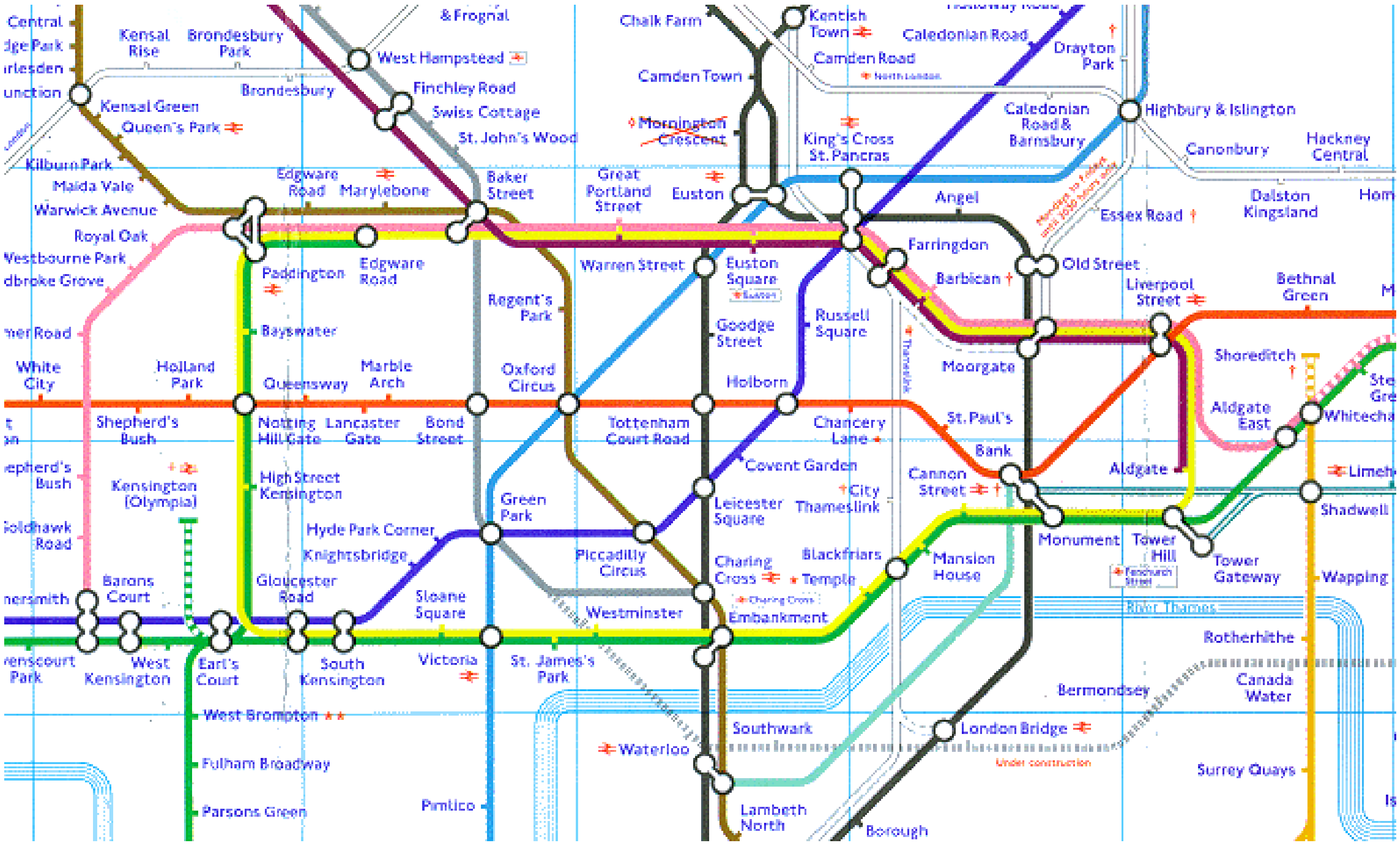
N.B.

Vertices Are Denoted By :

 Numbers : 1, 2, 3, …

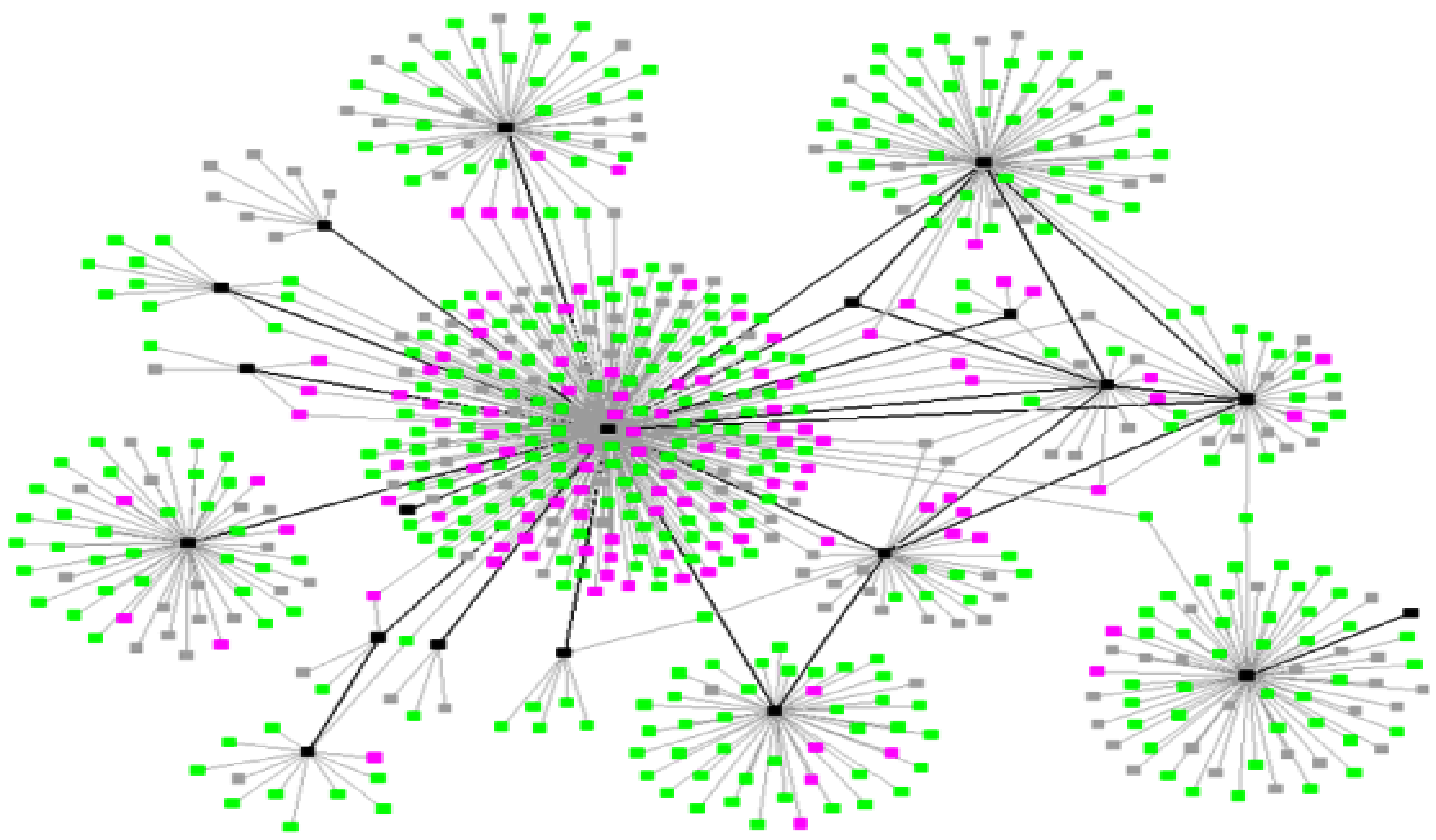
 Letters : a, b, c, v1, v2, …

Practical Examples of Graphs



Train Maps

Practical Examples of Graphs

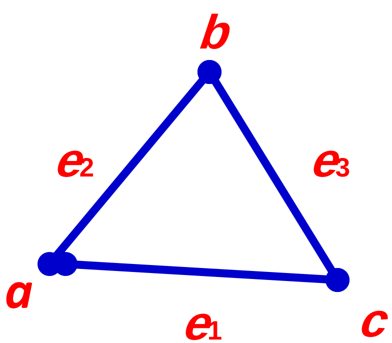


Social Networks

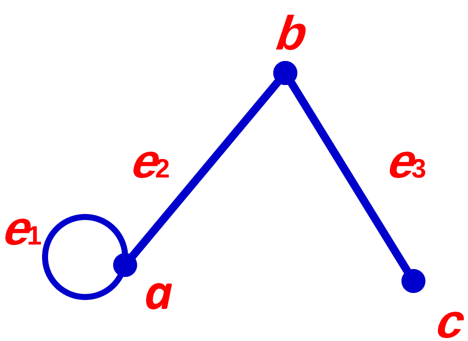
A Simple Graph?

- A simple graph contains no loop or parallel edges
  - No loop ⇛ Each edge connects two different vertices
  - No parallel ⇛ No two edges connect the same pair of vertices

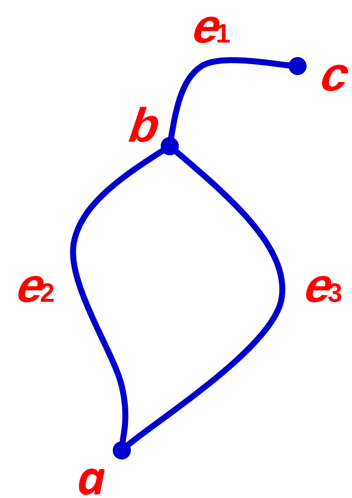
- Example of simple graph and not a simple graph



simple graph



not a simple graph

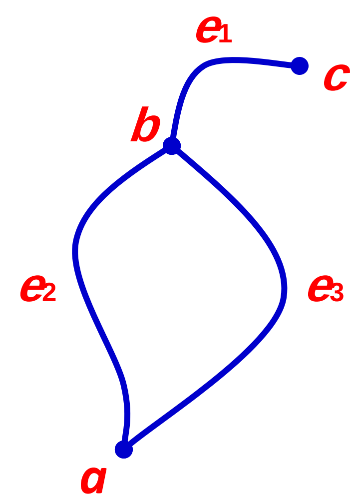


not a simple graph

Multigraphs & Psuedographs

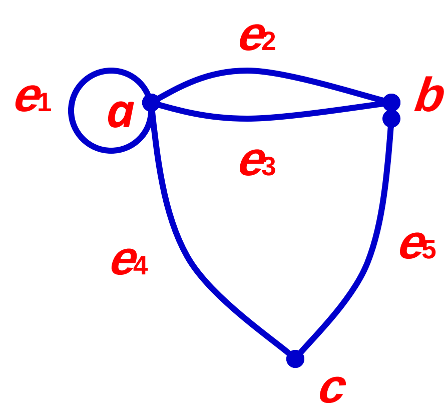
▶ Multigraphs

- Graph that may have multiple edges connecting the same vertices.



▶ Psuedographs

- Graph that may include loops, and possibly multiple edge connecting the same vertices.



A Directed Graph

- A graph directed (or digraph) (*V*, *E*) consists of
  - *V* – a nonempty set of Vertices (or nodes) ⇛ represents by point
  - *E* – a set of directed Edges ⇛ represents by line segments or curve
    - Each directed edge associated with an ordered pair of vertices (*u*, *v*), which said to start at *u* and end at *v*.

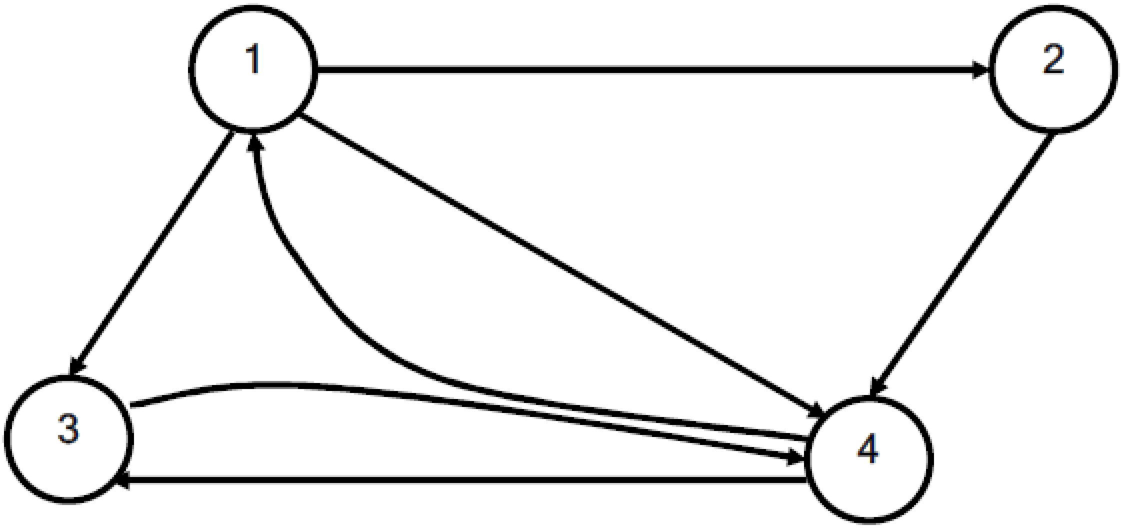
- Hence:

  A digraph is a directed one which consists of a finite set of points, called vertices, together with a finite set of directed edges, each of which joins a pair of distinct vertices. Thus a digraph contains no loops. Moreover, the directed edge $P_i\ P_j$ is different from the edge $P_j\ P_i$. Hence, a digraph G = G(E, V) is a graph in which each edge "e = (i, j)" has a direction from its "initial point i" to its "terminal point j". Two edges connecting the same two points "i, j" are permitted provided that they have opposite directions, i.e. they are (i, j) and (j, i).

A Directed Graph

- Simple directed graph – a directed graph with no loops and no multiple directed edges
- Directed multigraph – a directed graph with multiple directed edge
- Mixed graph – a graph with both directed and undirected edges

- **Example of a directed graph**



A Directed Graph

Directed Graphs Are Essential In Important Applications, such As:

- Pipelines networks .
- Producer-consumer relations .
- Traffic nets of one-way streets .
- Flows of computations in a computer .
- Sequences of jobs in construction work .

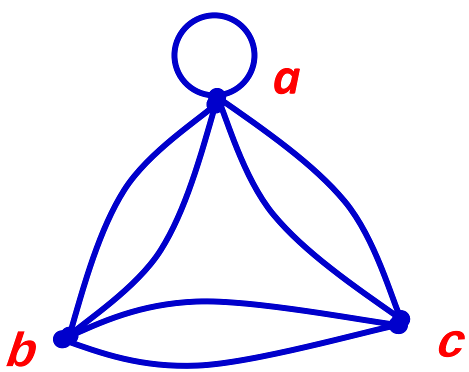
Summary: Types of Graph & its Structure

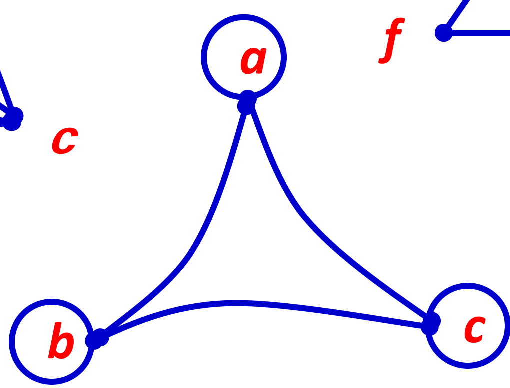
Type	Edges	Multiple Edges Allowed?	Loops Allowed?
Simple Graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Psuedograph	Undirected	Yes	Yes
Simple Directed Graph	Directed	No	No
Directed Multigraph	Directed	Yes	Yes
Mixed Graph	Directed and Undirected	Yes	Yes

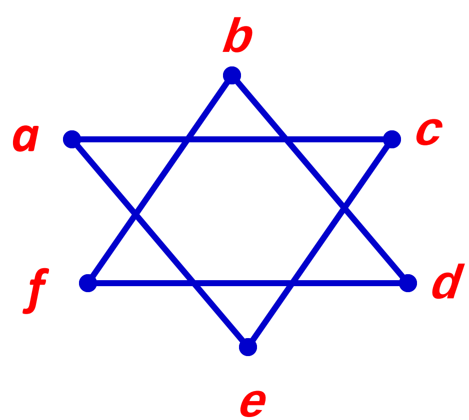
Check Point

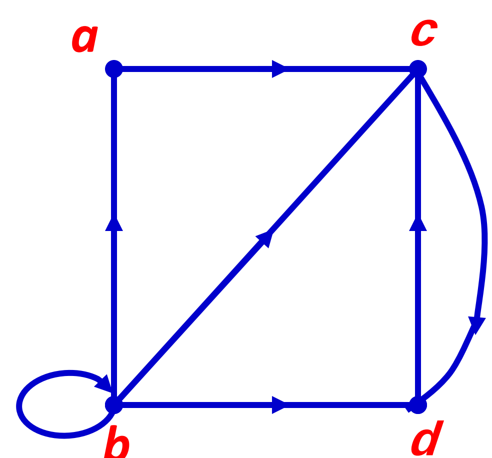
For each of the following graph,

- Determine the type of graph
- Find the number of vertices and edges
- If the graph is a not simple undirected graph, find a set of edges to remove to make it simple graph.









How to Graph a Model

- When we build a graph model, we need to make sure that we have correctly answered three key questions about the structure of a graph as following:

1. Are the edges of graph undirected or directed or both?

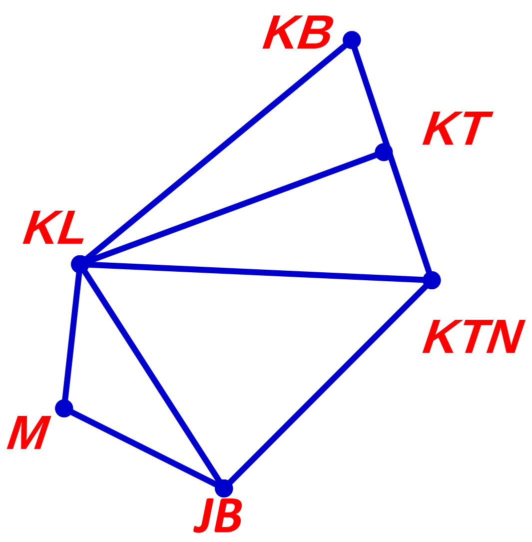
2. If the graph is undirected, are multiple edges present that connect the same pair of vertices? If the graph is directed, are multiple directed edges presents?

3. Are loops present?

Examples of Graph Models
Modeling Relationships Using Graphs

I. Modeling Computer Networks Using Graphs

- A network is made up of data centers (represents the location by point) and communication links between computer (represents the links by line segments).



II. Modeling Relationships Using Graphs

Modeling Konigsberg With A Graph

- The first paper in Graph Theory was written by Euler in 1736 when he settled the famous unsolved problem of his day, known as the *Konigsberg Bridge Problem* . Konigsberg (55.2° North lattitude and 22° East longitude) is now called Kaliningrad and is in Lithuania which recently separated from U.S.S.R.

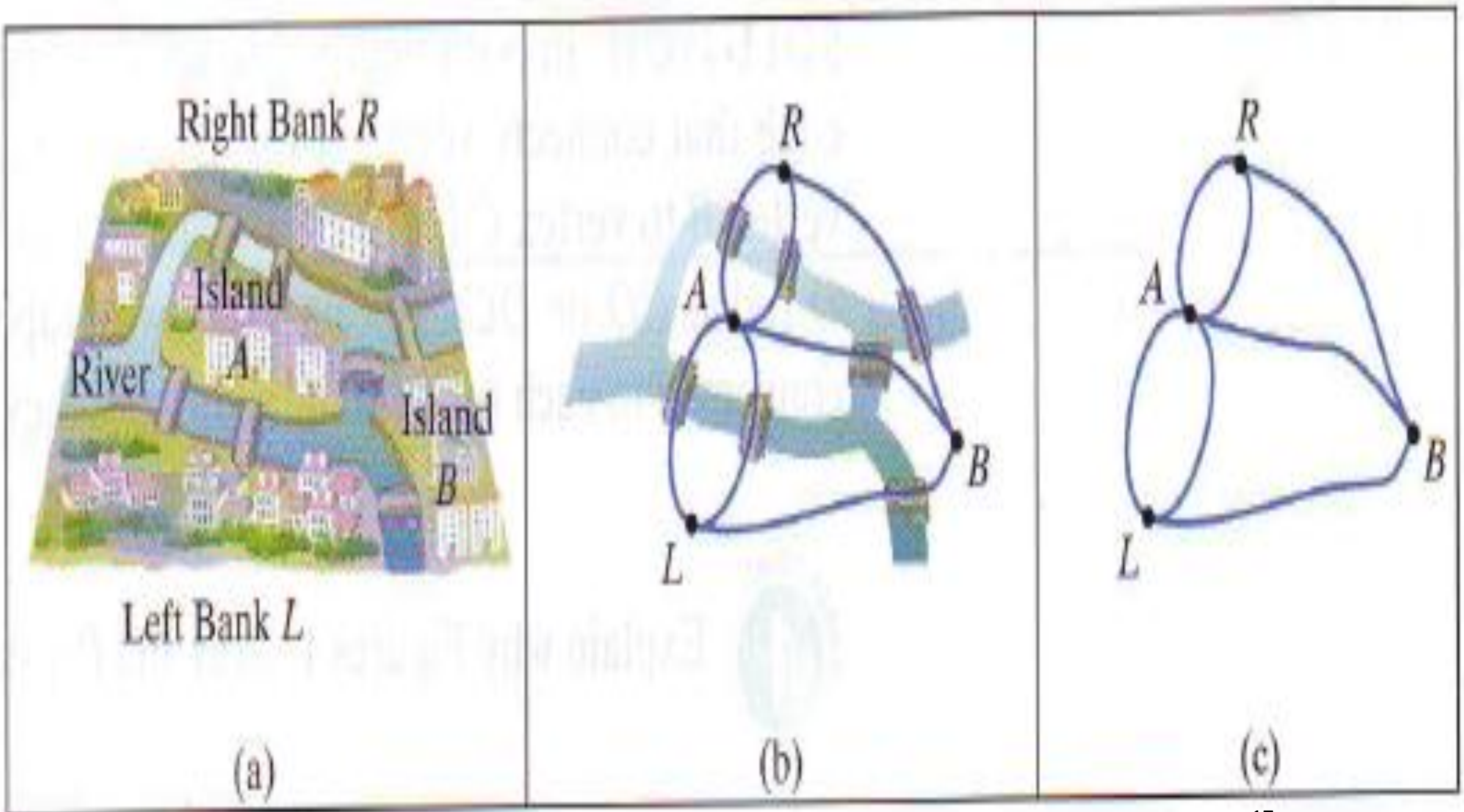
- The two islands "A" and "B" and the two banks "L" and "R" of the Pragel river are connected by seven bridges as shown above. A The problem was to start from any one of the land areas, walk across each bridge exactly once and return to the starting point.

- Euler proved that this problem has no solution. Euler abstracted the problem by replacing each land area by a point and each bridge by a line joining these points.

- The Konigsberg bridge problem is the same as the problem of drawing the graph representing it without lifting the pen from the paper and without retracing any line and coming back to the starting point.

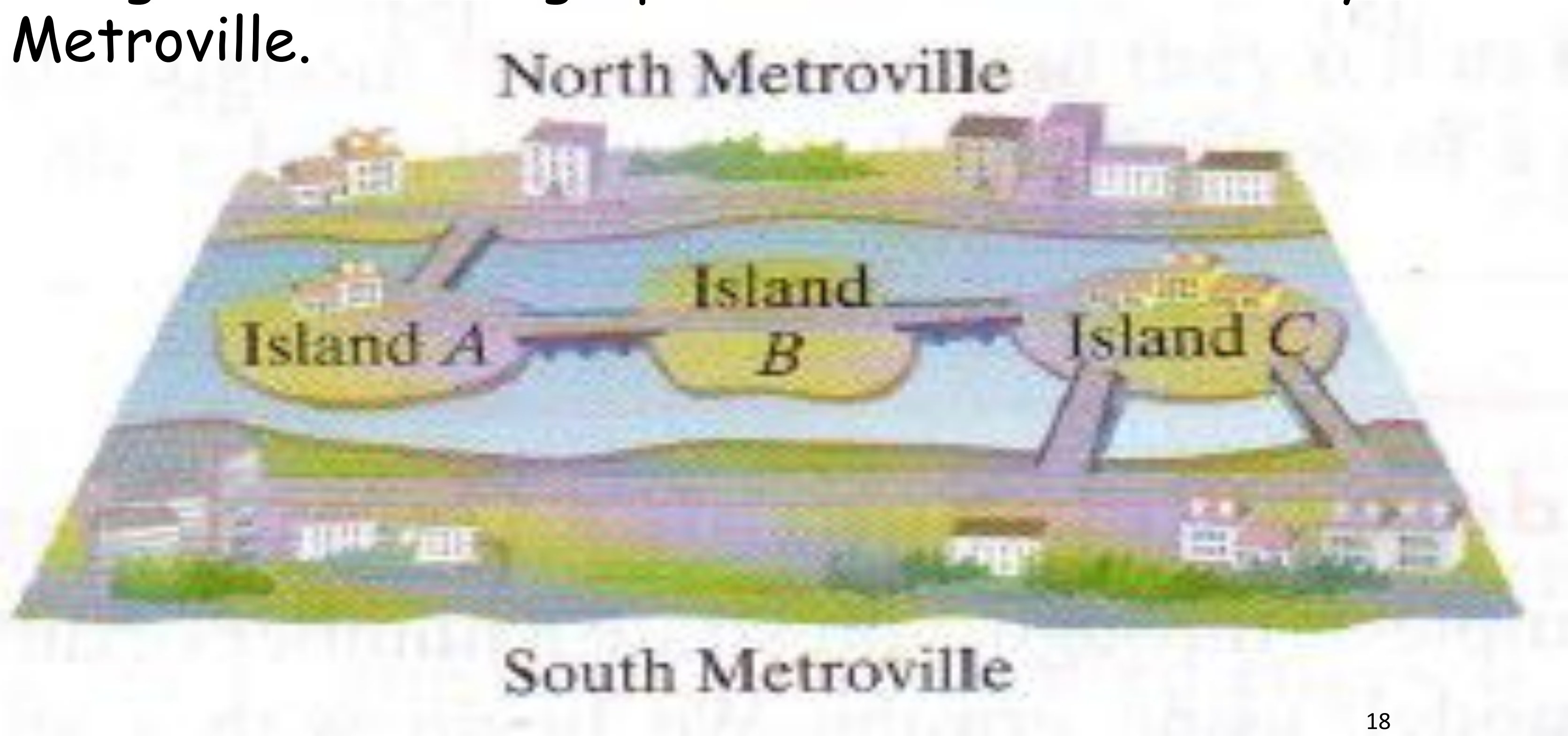
II. Modeling Relationships Using Graphs

Modeling Konigsberg With A Graph



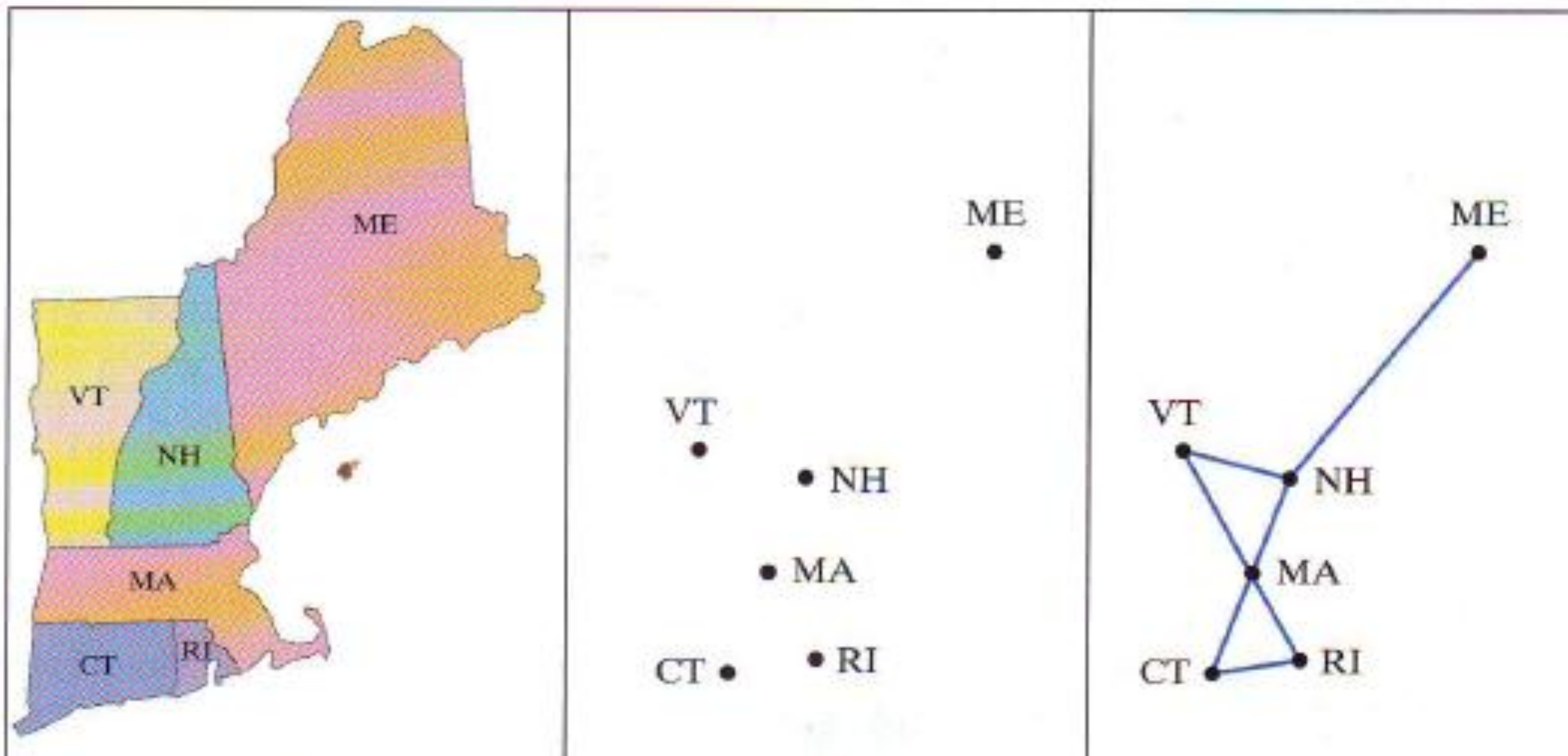
Check Point

The city of Metroville is located on both banks and three islands of the Metro River. The Figure show down that the town's sections are connected by five bridges. Draw a graph that models the layout of Metroville.



III. Modeling Bordering Relationships For the New England States

The map in Figure (a) shows the New England states. Figures (b) & (c) show a graph that models which New England states share a common border. Vertices to are used to represent the states and edges to represent common borders.



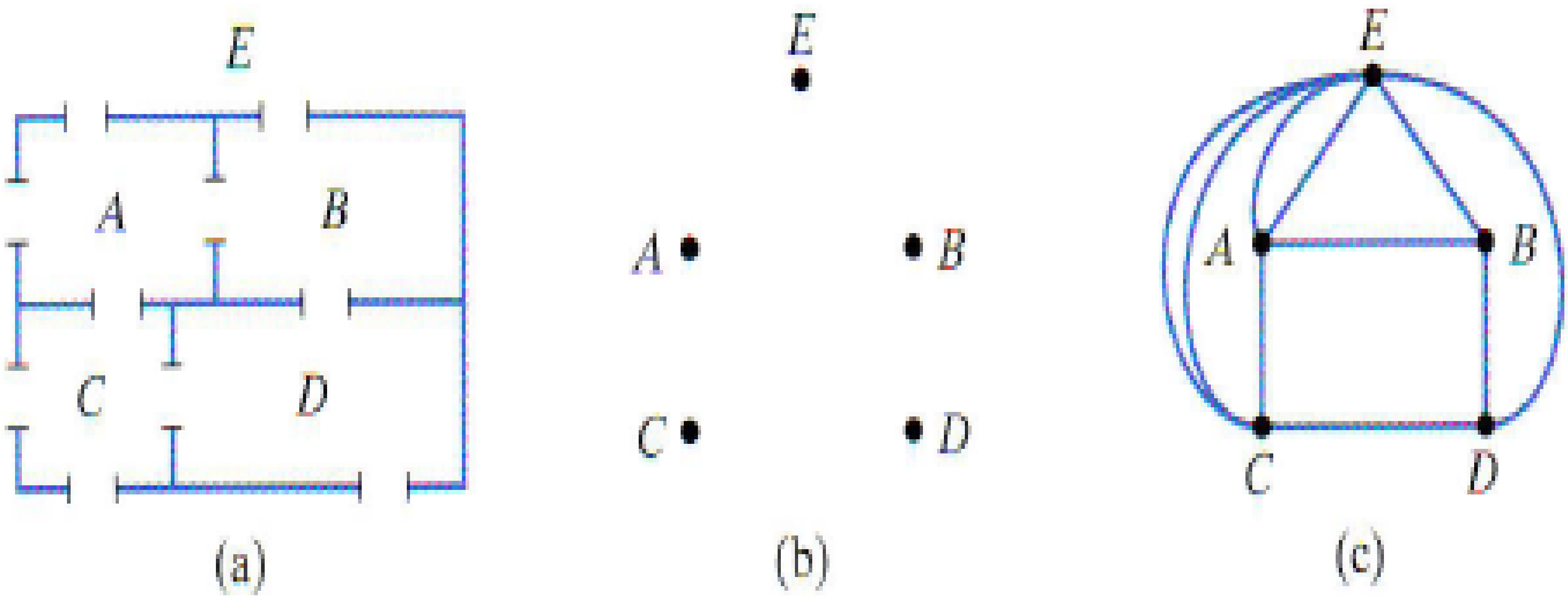
(a) (b) (c)

Check Point

- Create a graph that models the bordering relationships among the five states shown below.

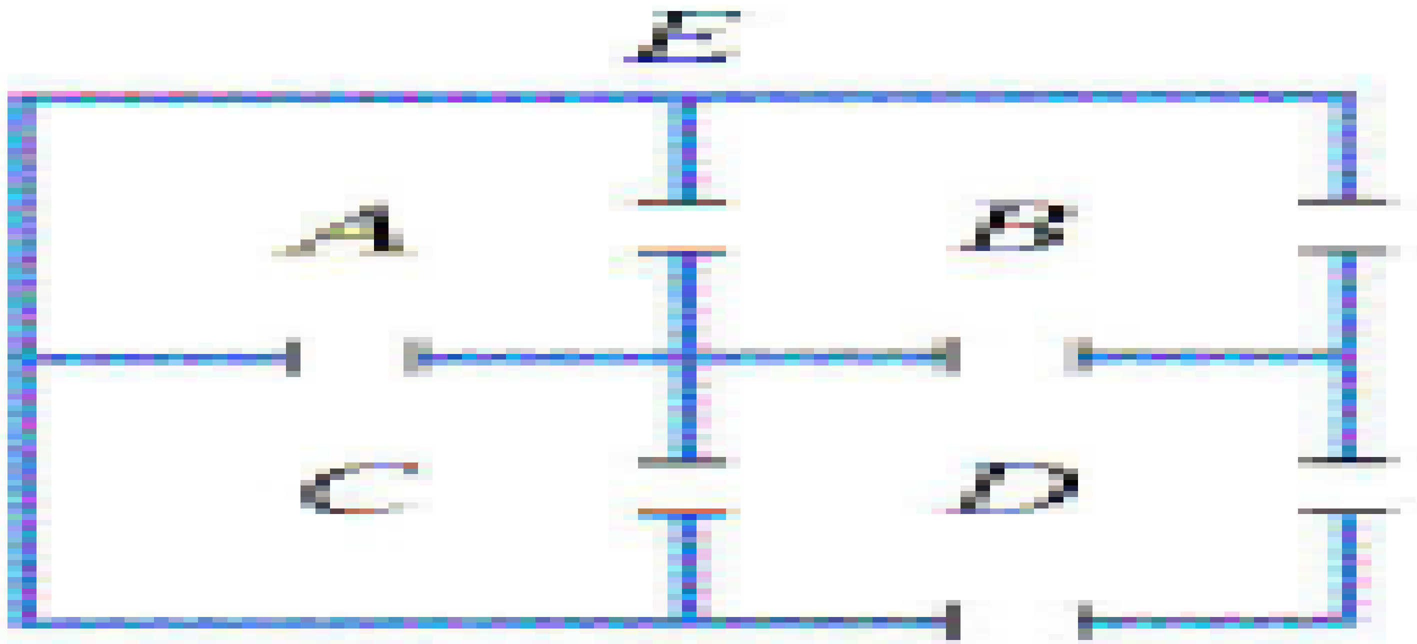
IV. Modeling Connecting Relationships In A Floor Plan

The floor plan of a four-room house is shown in figure (a). The rooms are labeled A, B, C, and D. The outside of the house is labeled E. The openings represent doors. A graph that models the connecting relationships in the floor plan is given in figure (c). Vertices represent the rooms and the outside. Edges represent the connecting doors.



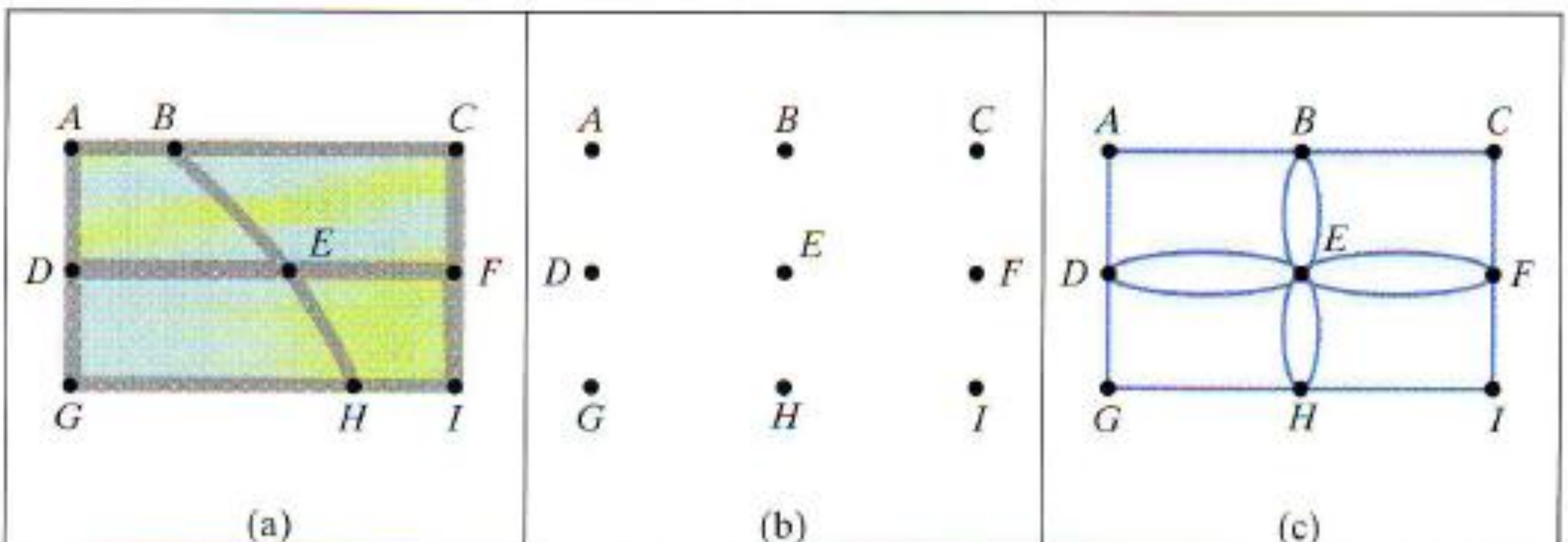
Check Point

- The floor plan of a four-room house is shown below. The rooms are labeled A , B , C, and D .The outside of the house is labeled E. Draw a graph that models the connecting relationships in the floor plan.



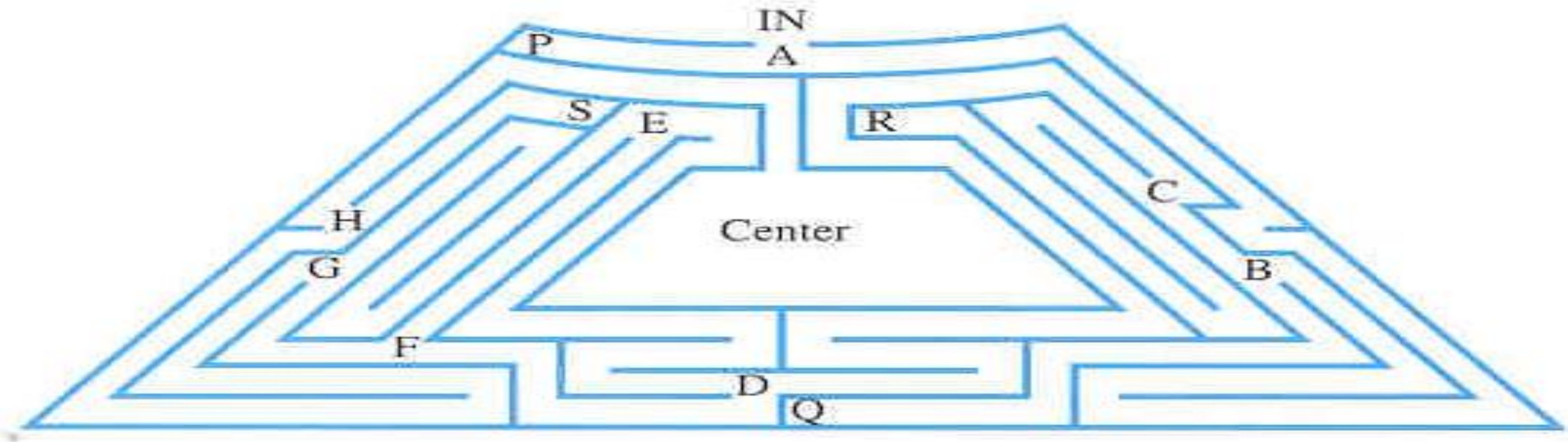
V. Modeling Walking Relationships For A Neighbourhood's Streets

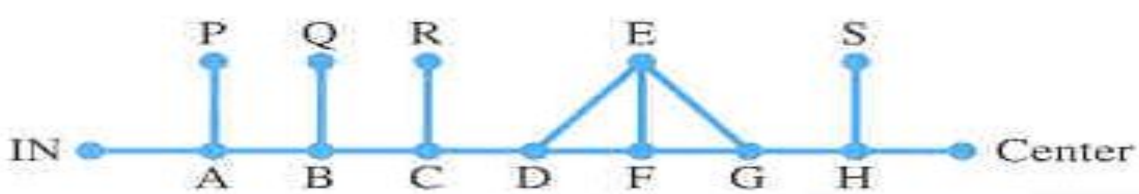
• A mail carrier delivers mail to the four-block neighborhood shown in Figure (a). He parks his truck at the intersection shown in the figure and then walks to deliver mail to each of the houses. The streets on the outside of the neighborhood have houses on one side only. By contrast, the interior streets have houses on both sides of the street. On these streets, the mail carrier must walk down the street twice, covering each side separately. Figure (c) shows a graph that models the streets of the neighborhood walked by the mail carrier. Vertices are used to represent the street intersections and corners. One edge is used if streets must be covered only once and two edges for streets that must be covered twice.



VI. Modeling The Structure of Mazes متاهات

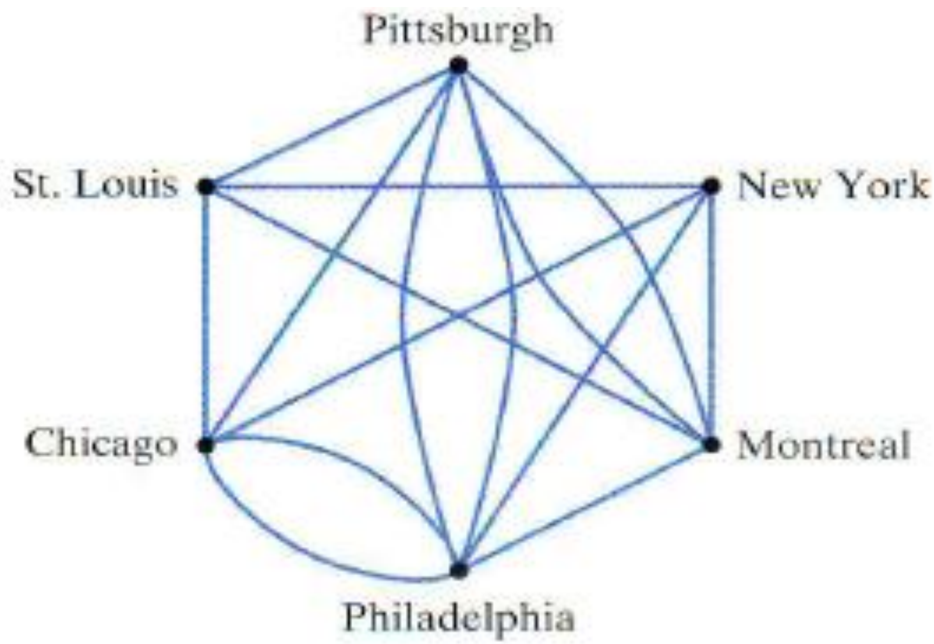
• Graphs can be used to clarify the structure of mazes. Vertices represent entrances to the maze and points in the maze where there is either a dead end or a choice of two or more directions to proceed. Edges show how these points are connected. For example, here is the 1690 design for the hedge maze at Hampton Court in England and a graph that clarifies its structure.





Exercise Set 1.

1. The graph models the baseball schedule for a week. The vertices represent the teams. Each game played during that week is represented as an edge between two teams. Use the information in the graph to solve the following Exercises.



i. How many games are scheduled for Pittsburgh during the week? List the teams that they are playing. How many times are they playing each of these teams?
ii. How many games are scheduled for Montreal during the week? List the teams that they are playing. How many times are they playing each of these teams?
iii. Do the positions of New York and Montreal correspond to their geographic locations on a map? If not, is the graph drawn incorrectly? Explain y our answer .

Exercise Set 1.

2. In the following exercises, draw two equivalent graphs for each description.

a. The vertices are A ,B,C, and D .The edges are AB, BC, BD, CD and CC.

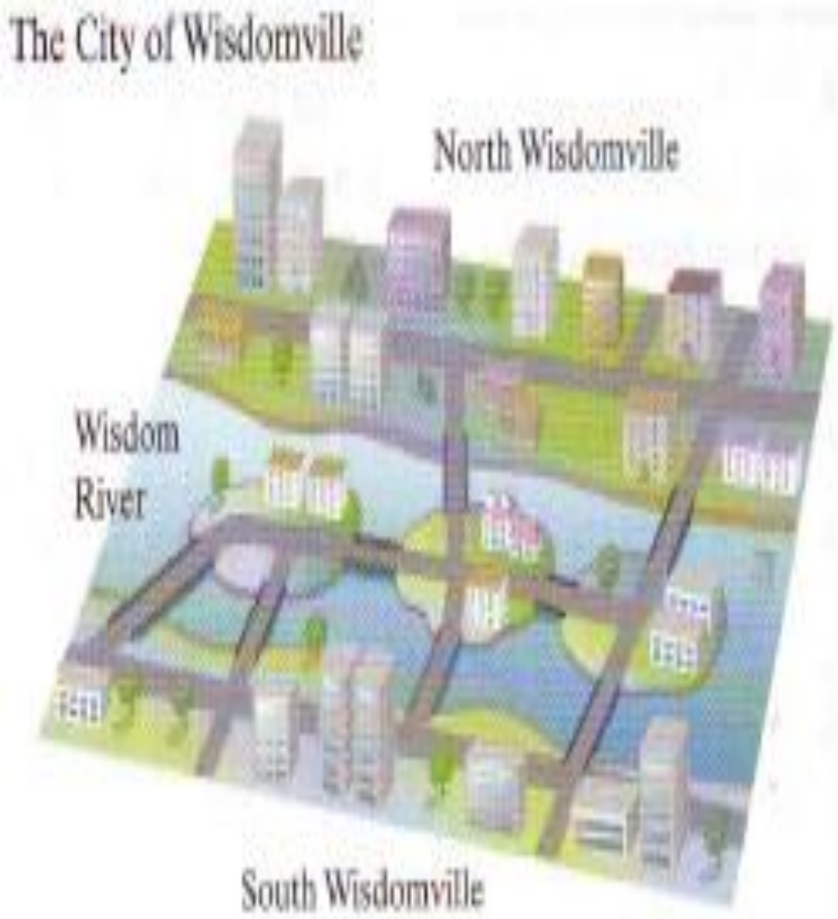
b. The vertices are A, B, C, and D. The edges are AD, BC, DC, BB, and DB.

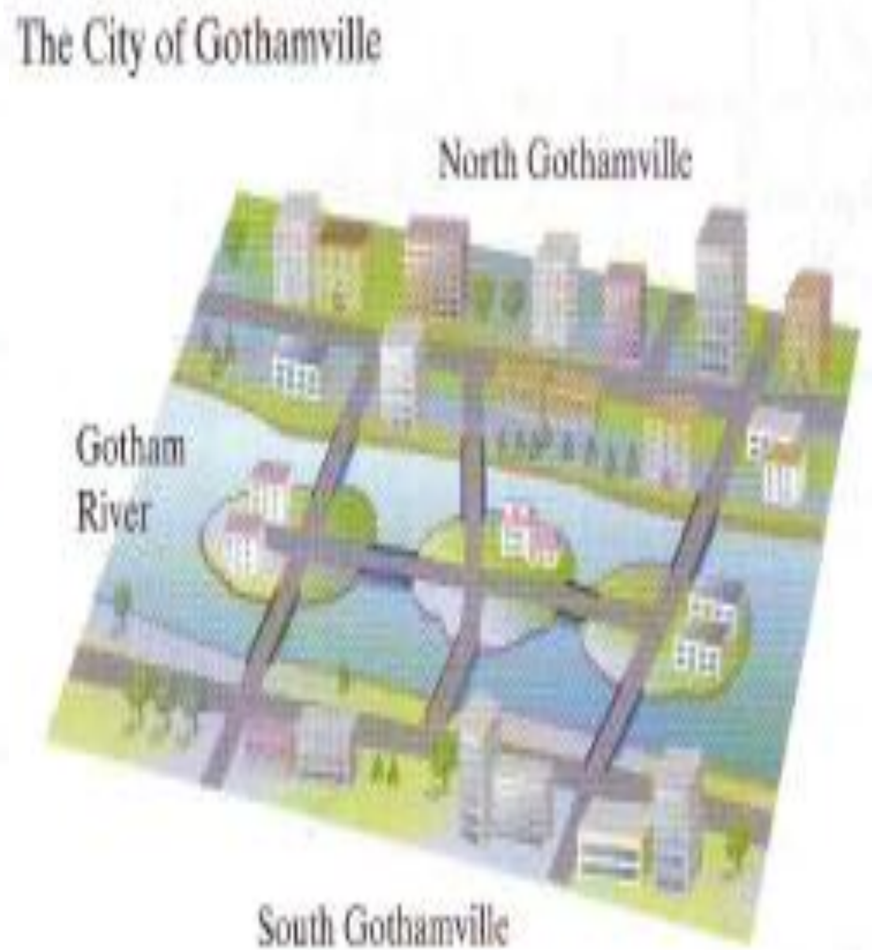
3. Eight students form a math homework group. The students in the group are Zeb, Stryder, Amy, Jed, Evito, Moray, Carrie, and Oryan. Prior to forming the group, Stryder was friends with everyone but Moray. Moray was friends with Zeb, Amy, Carrie, and Evito. Jed was friends with Stryder, Evito, Oryan, and Zeb. Draw a graph that models pairs of friendships among the eight students prior to forming the math homework group.

4. An environmental action group has six members A, , B, C, D, E, and F. The group has three committees: The Preserving Open Space Committee (B, D, and F), the Fund Raising Committee (B, C, and D), and the Wetlands Protection Committee (A, C, D, and E). Draw a graph that models the common members among committees. Use vertices to represent committees and edges to represent common members.

Exercise Set 1.

5. Draw a graph that models the layout of the cities shown in maps i-ii. Use vertices to represent the land masses and edges to represent the bridges.



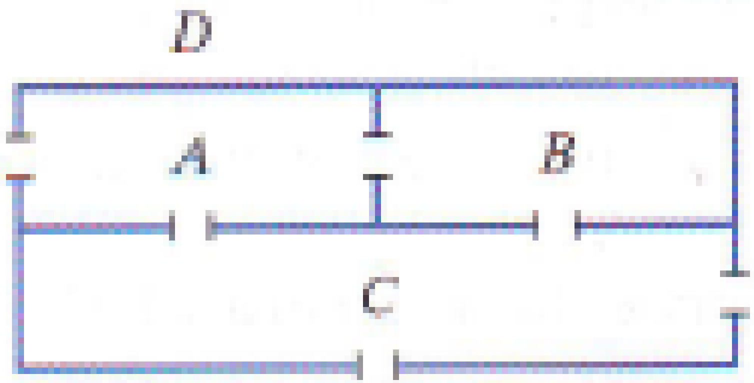


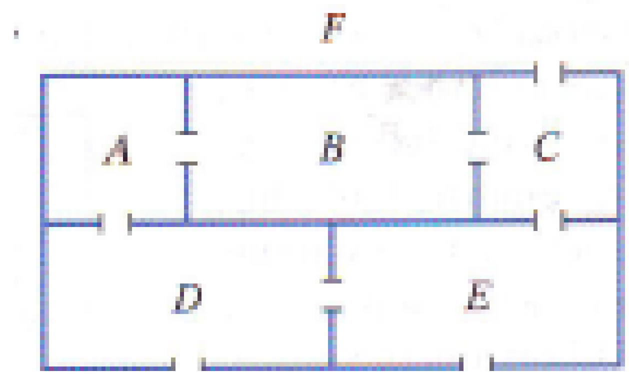
Exercise Set 1.

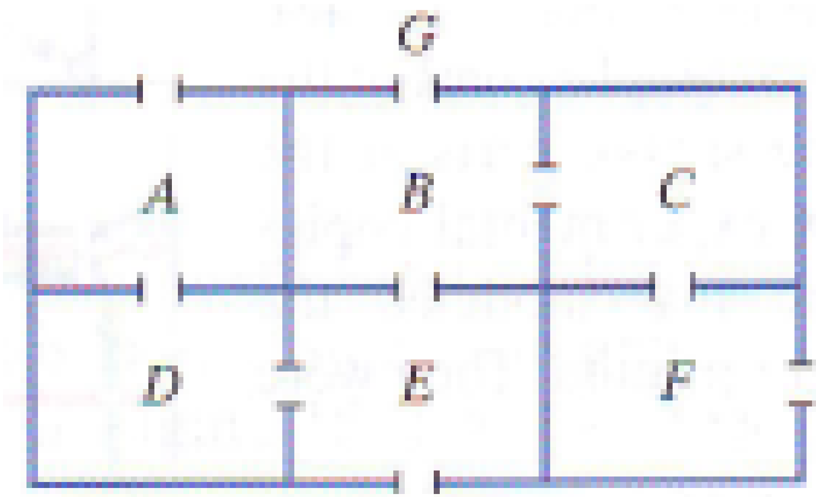
6. In the following Exercises , create a graph that models the bordering relationships among the states shown in each map. Use vertices to represent he states and edges to represent common borders.

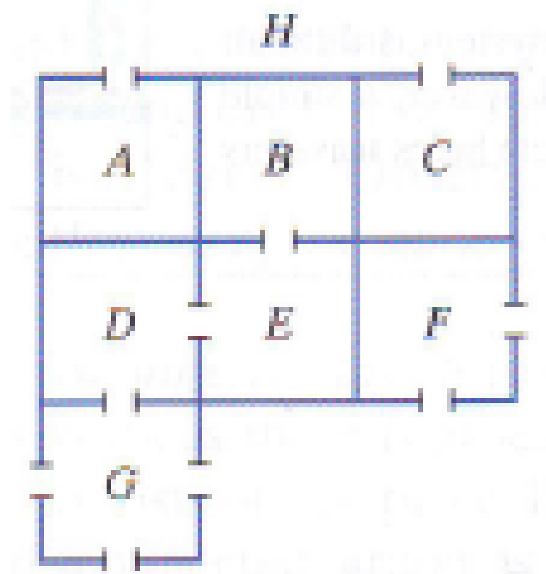
Exercise Set 1.

7. In Exercises 15-18, draw a graph that models the connecting relationships in each floor plan. Use vertices to represent the rooms and the outside, and edges to represent he connecting doors.





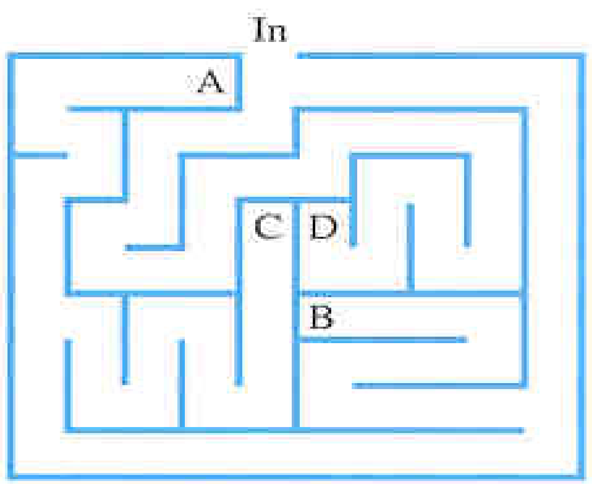




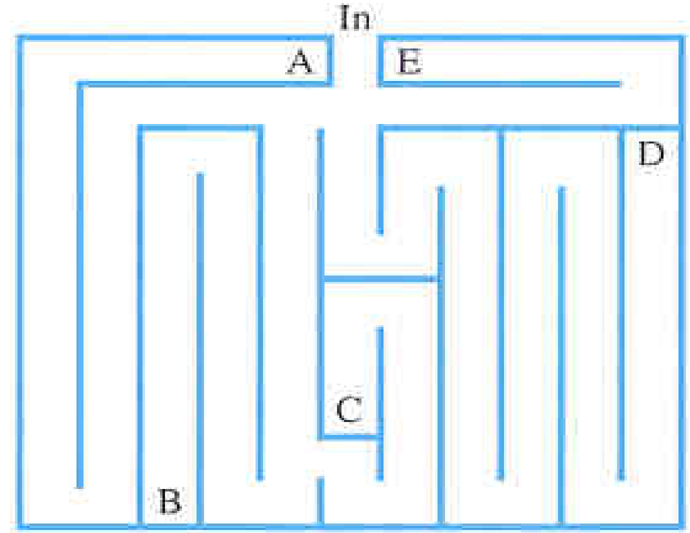
Exercise Set 1.

8. Draw a graph to clarify the underlying structure of each of the following mazes. Points that are entrances or dead ends have been labeled, but you will need to label points where there is a choice of two or more directions to proceed

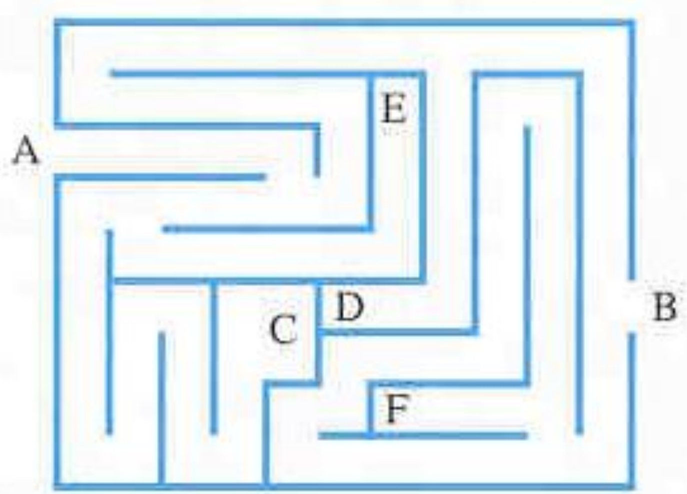
(a)



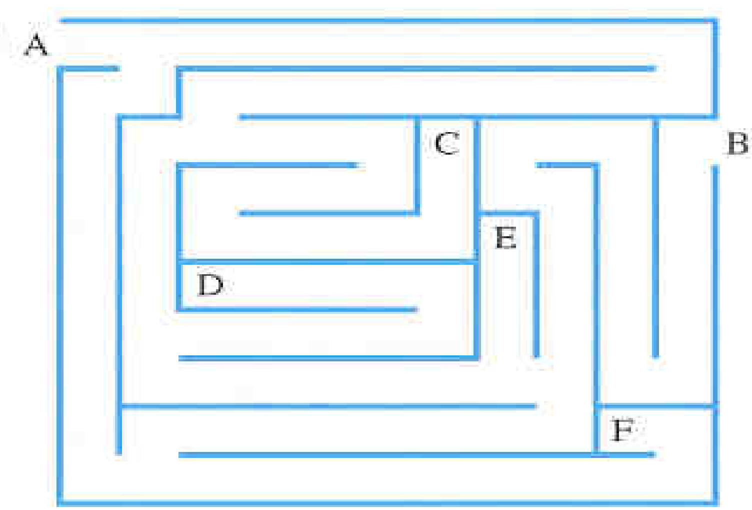
(b)



(c)



(d)

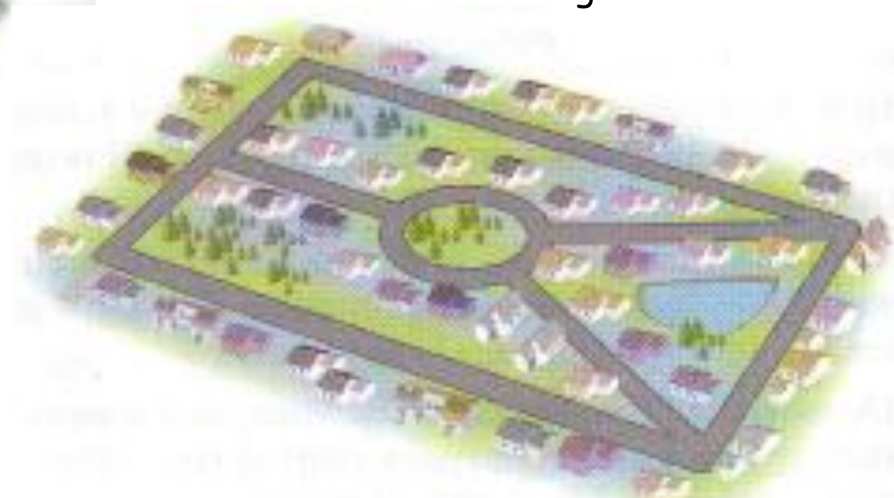


Exercise Set 1.

9. In Exercises 9.a – 9.b, a security guard needs to walk the streets of the neighborhood shown in each figure. The guard is to walk down each street once, whether or not the street has houses on both sides. Draw a graph that models the neighborhood. Use vertices to represent the street intersections and corners. Use edges to represent the streets the security guard needs to walk.

Fig. 9.a

Fig. 9.b



2. Graph Terminology

- Introduce the basic terminology of graph theory
- Introduce several classes of simple graph
- Introduce subgraph

Basic Graph Terminology

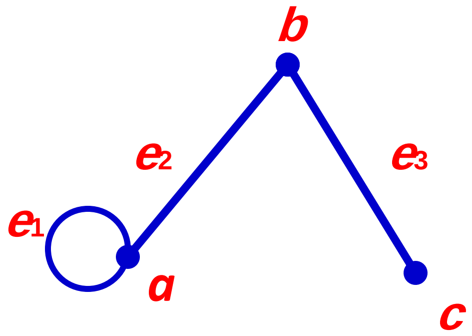
- Adjacent Vertices and Incident/تابعة/واقعةEdges
- Degree: Isolated & Pendant معلق& Even & Odd
- Initial & Terminal Vertex
- Degree/In-Degree/Out-Degree
- Walks & Paths & Circuits
- Connected & Disconnected Graphs
- Bridge/cut edge
- Cut vertex

Adjacent متجاور Vertices and Incident متواجد Edges

- Two vertices *u* and *v* in an undirected graph *G* are called **adjacent** (or neighbors) in *G* if *u* and *v* are endpoints of an edge of *G*. (or if there is at least one edge connecting them). It is helpful to think of adjacent vertices as connected vertices

- If the edge *e* is associated with $\{u, v\}$, *e* is called **incident** with the vertices *u* and *v*. (*e* connects *u* and *v*). Also, The vertices u and v are incident متواجدتان with the edge e.

- <u>Example</u>:
  - *a* and *b* are adjacent.
  - *a* and *c* not adjacent.
  - e_2 is incident with *a* and *b* .

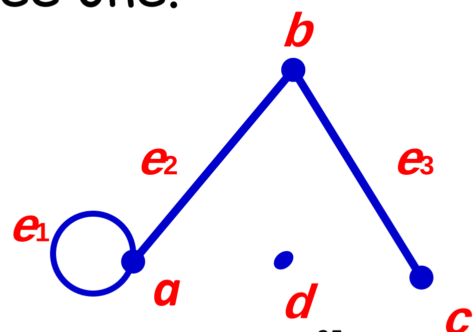


2. Graph Terminology

Degree, Isolated معزول and Pendant عالق Vertices

- <u>For Undirected Graph</u>

- The degree of a vertex in an undirected graph is the number of edges incident المتواجدة with it.
- A loop at a vertex contributes twice to the degree of that vertex (since it connects the vertex to itself, that loop contributes 2 to the degree of the vertex)
- The degree of the vertex *v* is denoted by deg (*v*).
- A vertex of degree zero is called isolated. (not adjacent to any vertex)
- A vertex is pendant/معلقة iff it has degree one.
- **Example:**
  - deg (*a*) = 3 , deg (*b*) = 2
  - deg (*c*) = 1 , so vertex *c* is pendant
  - deg (*d*) = 0 , so vertex *d* is isolated



Even and Odd Degree

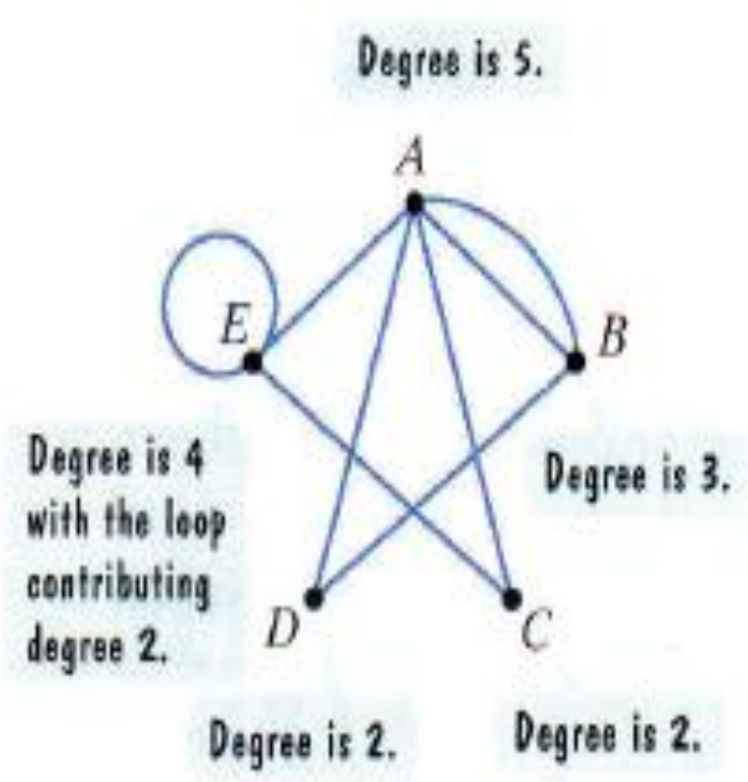
- <u>For Undirected Graph</u>

A vertex with an even number of edges attached to it is an even vertex.

For example, in the Figure shown in R.H.S., vertices E, D, and C are even.

A vertex with an odd Number of edges attached to it is an odd vertex.

In the Figure shown in R.H.S., vertices A and B are odd.



Check Point

For each of the following graph,

- Find the degree of each vertex
- Identify all the isolated and pendant vertices

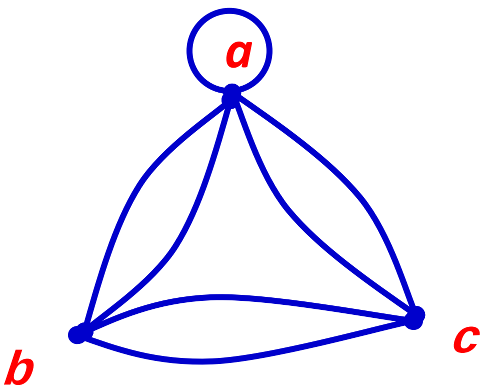


Fig. 1

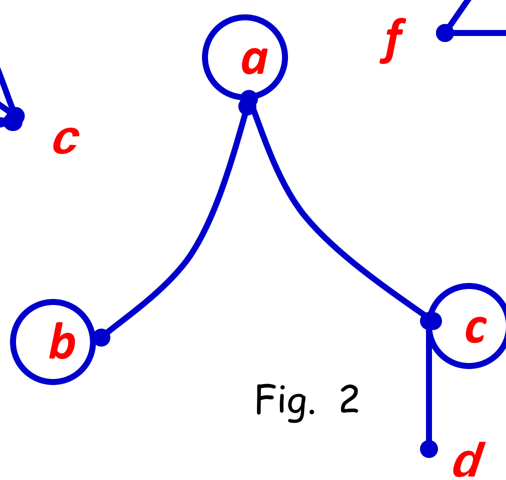


Fig. 2

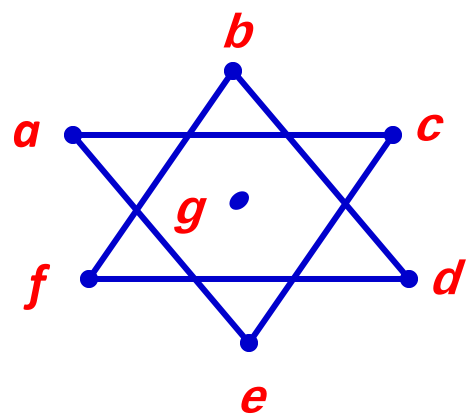
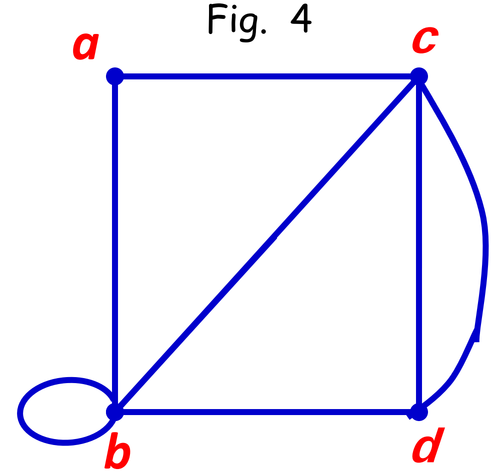


Fig. 3

Fig. 4



Theorem Involving Degrees

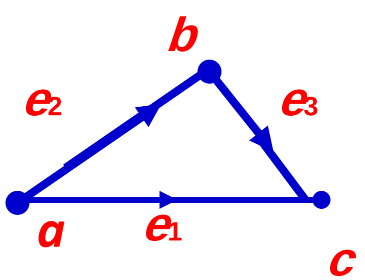
1. The Handshaking Theorem:
   - Let $G = (V, E)$ be an undirected graph with e edges. Then $$2e = \sum_{v \in V} \deg(v)$$
   - This theorem also applies if multiple edges and loops are present.
   - **Example:** How many edges are there in a graph with 10 vertices each of degree 6?

2. An undirected graph has an even number of vertices is of odd degree.

Initial and Terminal Vertex

For Undirected Graph

- When (u, v) is an edge of the graph G with a directed edge from the vertex u to the vertex v, u is said to be adjacent to v and v is said to be adjacent from u.
- The vertex u is called the initial vertex of (u, v), and v is called the terminal or end vertex of (u, v).
- The initial vertex and terminal vertex of a loop are the same.
- **Example**
  - a is adjacent to b
  - b is adjacent to c
  - c is adjacent from a (b)
  - Vertex a is initial vertex of (a, b)
  - Vertex b is terminal vertex of (a, b)



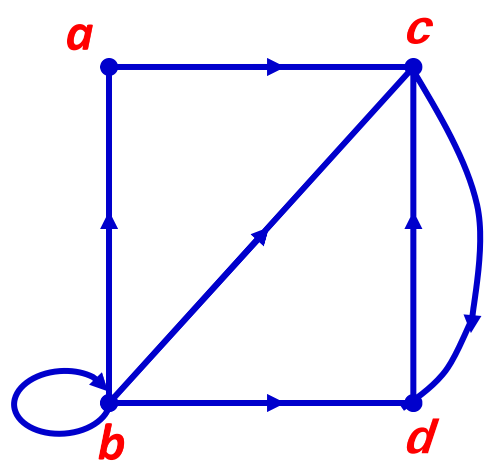
2. Graph Terminology

In-Degree and Out-Degree

For Undirected Graph

- For directed graph, the **in-degree** of a vertex *v*, denoted by $\deg^-(v)$, is the **number of edges with *v* as their terminal vertex**.
- For directed graph, the **out-degree** of a vertex *v*, denoted by $\deg^+(v)$, is the **number of edges with v as their initial vertex**.
- Loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.
- The summation: $\sum_{v\in V}\deg^-(v)=\sum_{v\in V}\deg^+(v)=|E|$ holds
- **Example:**
  - –The in-degrees in *G* are:
    - $\deg^-(a) = 1$, $\deg^-(b) = 1$
    - $\deg^-(c) = 3$, $\deg^-(d) = 2$
  - –The out-degrees in *G* are:
    - $\deg^+(a) = 1$, $\deg^+(b) = 4$
    - $\deg^+(c) = 1$, $\deg^+(d) = 1$

$$\sum_{v\in V}\deg^-(v)=\sum_{v\in V}\deg^+(v)=|E| = 7$$



Check Point

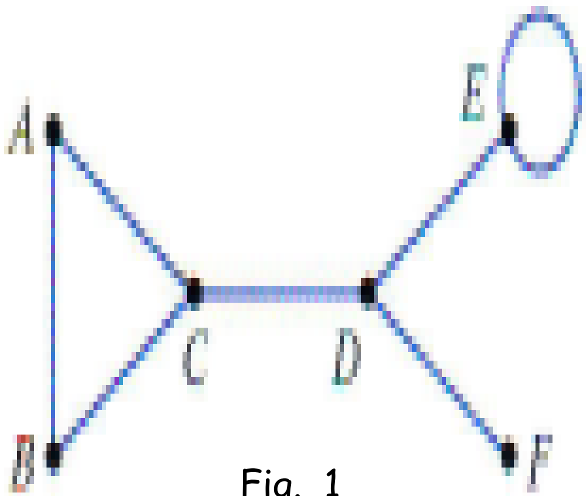


Fig. 1

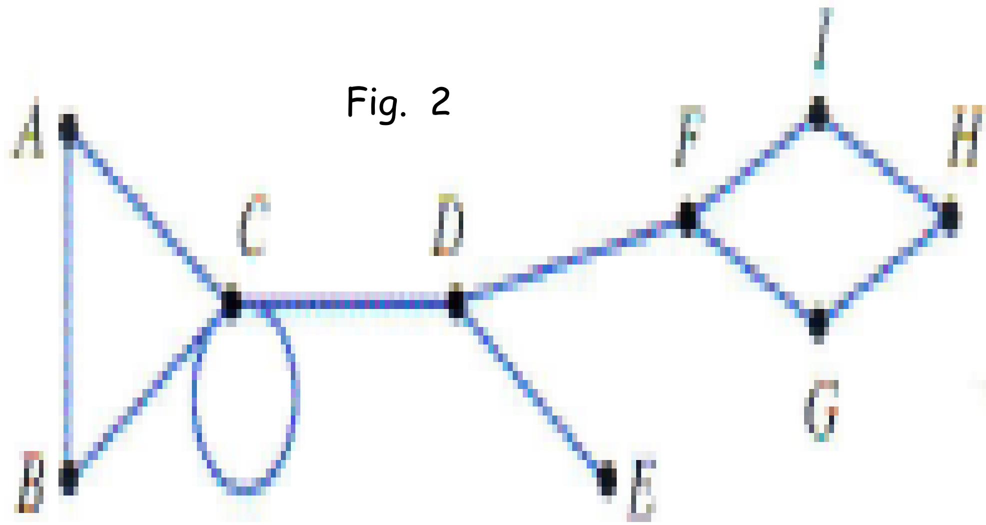


Fig. 2

Use each of the given two graphs to answer the following:

- Find the degree of each vertex in the graph.
- Identify the even vertices and identify the odd vertices.
- Which vertices are adjacent to vertex A and vertex E?
- Which vertices are adjacent o vertex D and vertex F?

Check Point

For each of the following directed graph, Find the in-degree and out-degree of each vertex

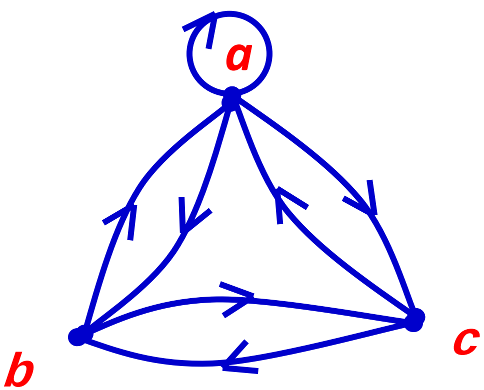


Fig. 1

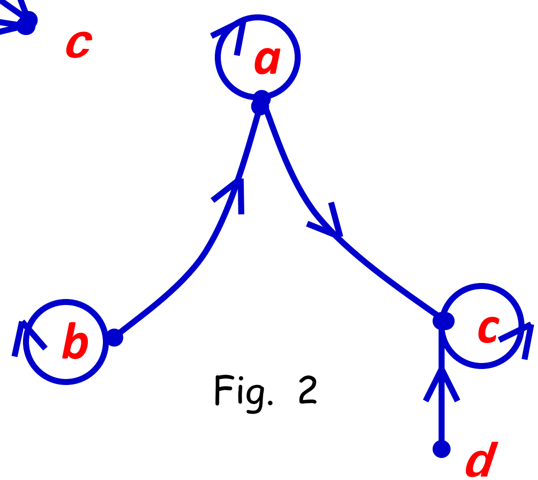


Fig. 2

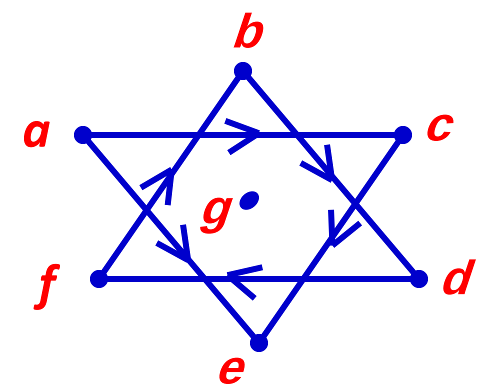


Fig. 3

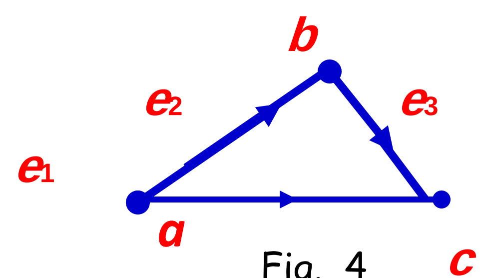


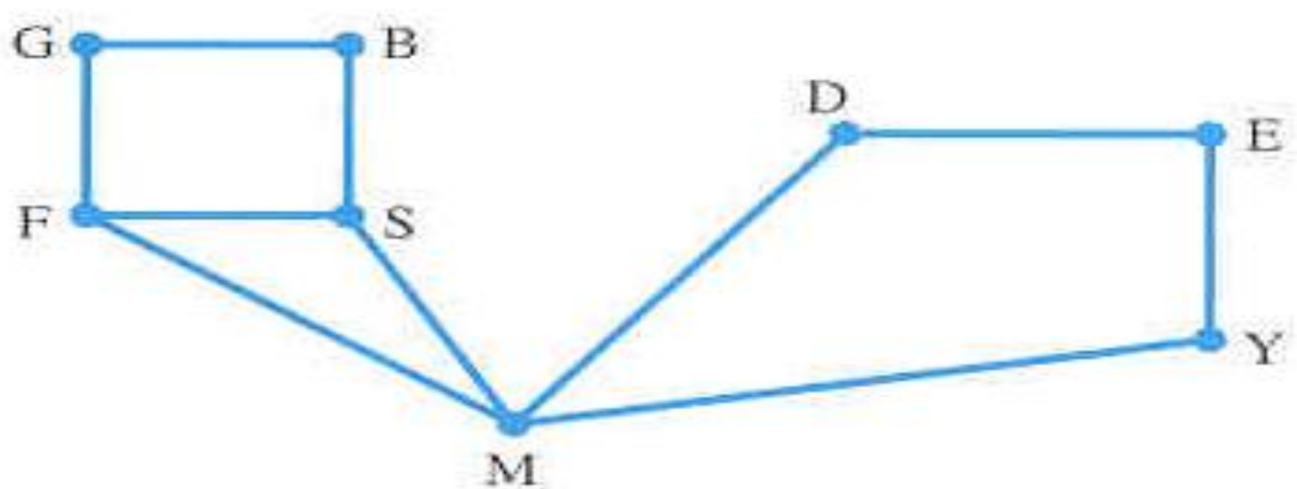
Fig. 4

Walks & Paths & Circuits

Walk:

A walk in a graph is a sequence of vertices, each linked to the next vertex by specified edge of the graph.

We can think of a walk as a route with a pencil without lifting the pencil from the graph. M → S → F → S → B and F → M → D → E are walks; M → D → S is not a walk, however since there is no edge between D and S.

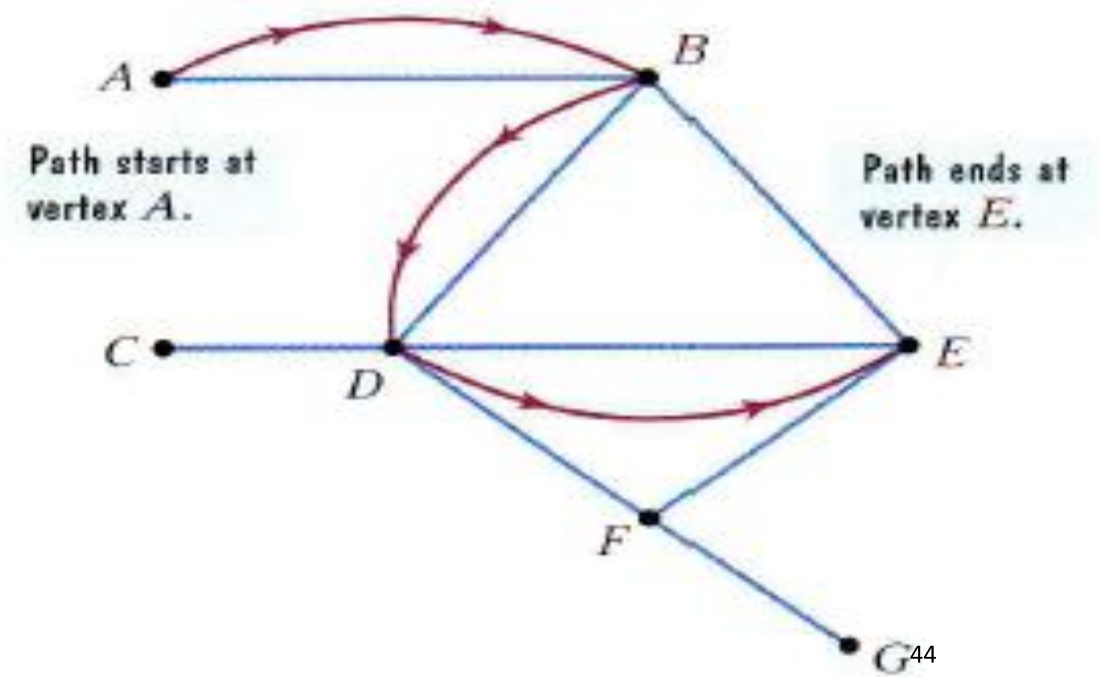


2. Graph Terminology

Walks & Paths & Circuits

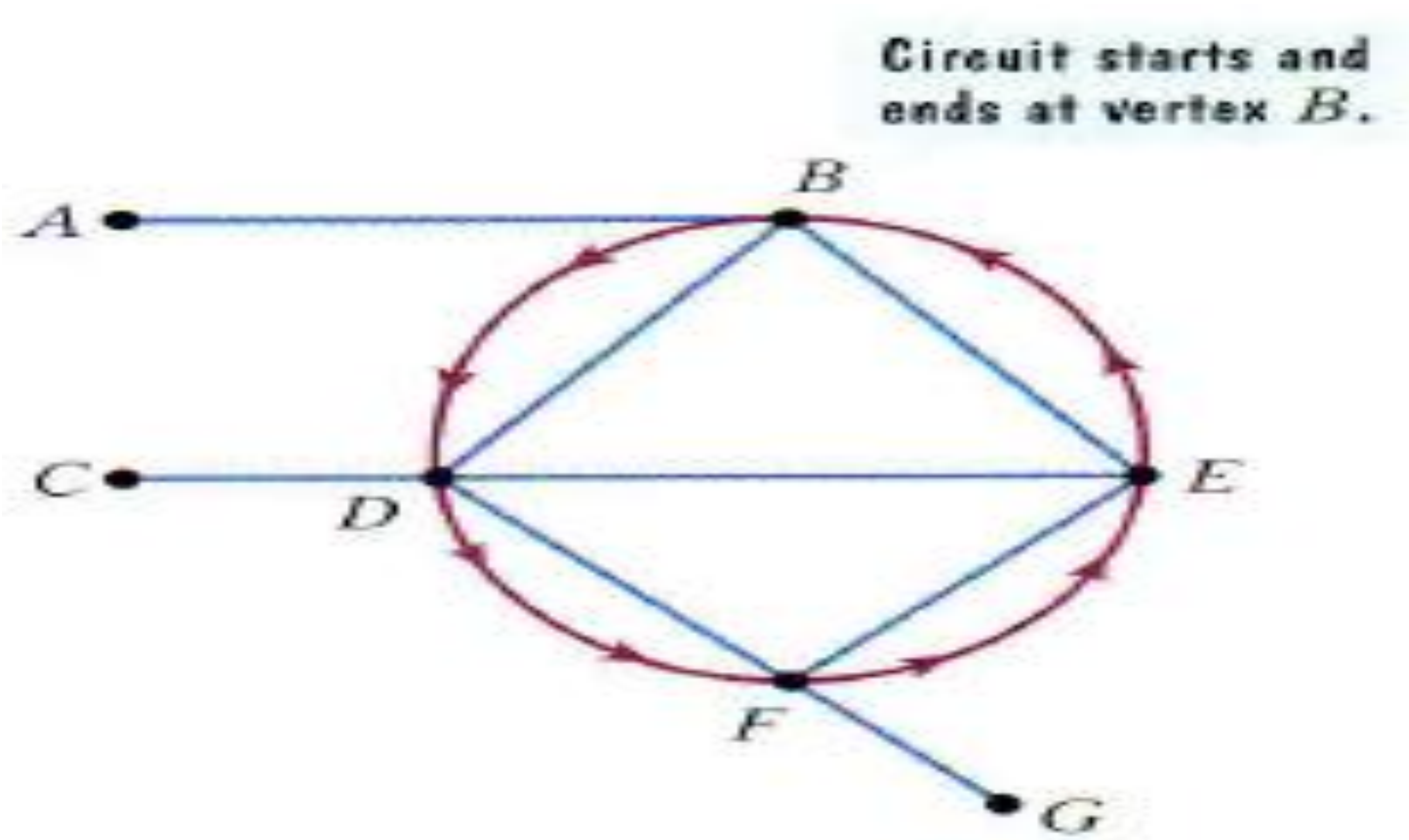
- A path in a graph is a sequence of adjacent vertices and the edges connecting them. Although a vertex can appeal on the path more than once, an edge can be part of a path only once. A path along the given graph, is shown in red. You can think of this path as movement from vertex A to vertex B to vertex D to vertex E. We can refer to this path using a sequence of vertices separated by commas. Thus, the path shown below is described b y A, B, D, E.

A path in a graph is a walk that uses no edge more than once.



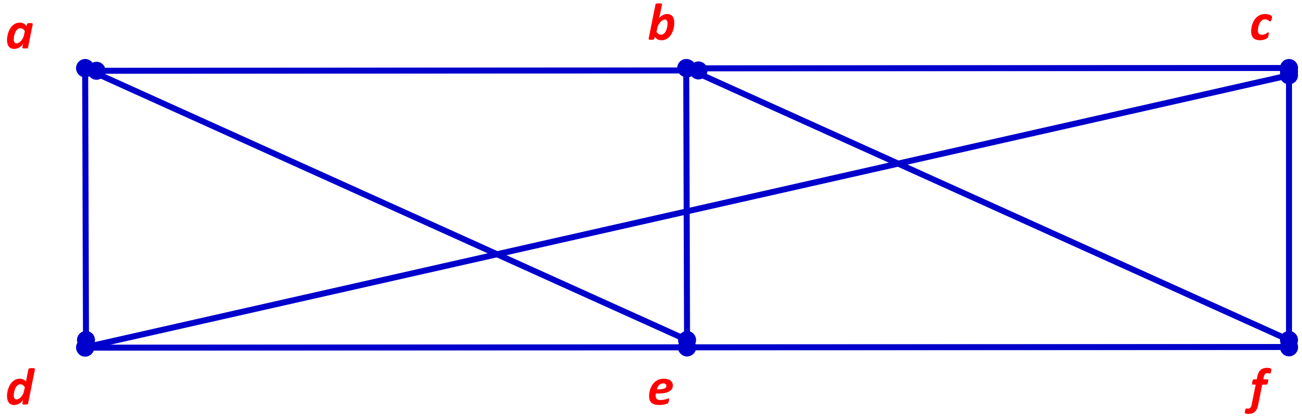
Walks & Paths & Circuits

A circuit is a path that begins and ends at the same vertex. In the figure shown below path given by B, D, F, E, B, is a circuit. Observe that every circuit is a path. However, because not every path ends at the same vertex where it starts, not every path is a circuit.



Walks & Paths & Circuits

- Example:
  - *a, d, c, f, e* is a path
  - *b, c, f, e, b* is a circuit
  - *d, e, c, a* is a not a path
  - *a, b, e, d, a, b* is not a path



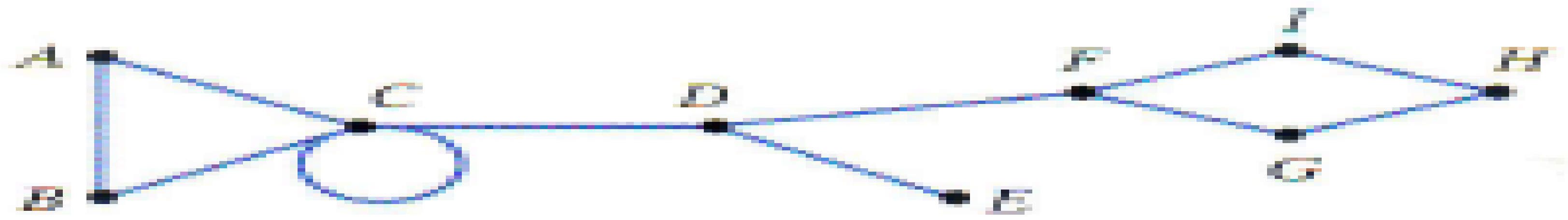
- Paths in an Acquaintanceship تعارف Graphs
  - There is a path between two people if there is a chain of people linking these people, where two people adjacent in the chain know one another.

Check Point

Use the given graph to answer the following:

1. Use vertices to describe two paths that start at vertex A and end at vertex D.
2. Use vertices to describe two paths that start at vertex B and end at vertex D.
3. Which edges shown on the graph are not included in the following path: E, E, D, C, B, A?
4. Which edges shown on the graph are not included in the following path: E, E, D,C,A, B?

Check Point

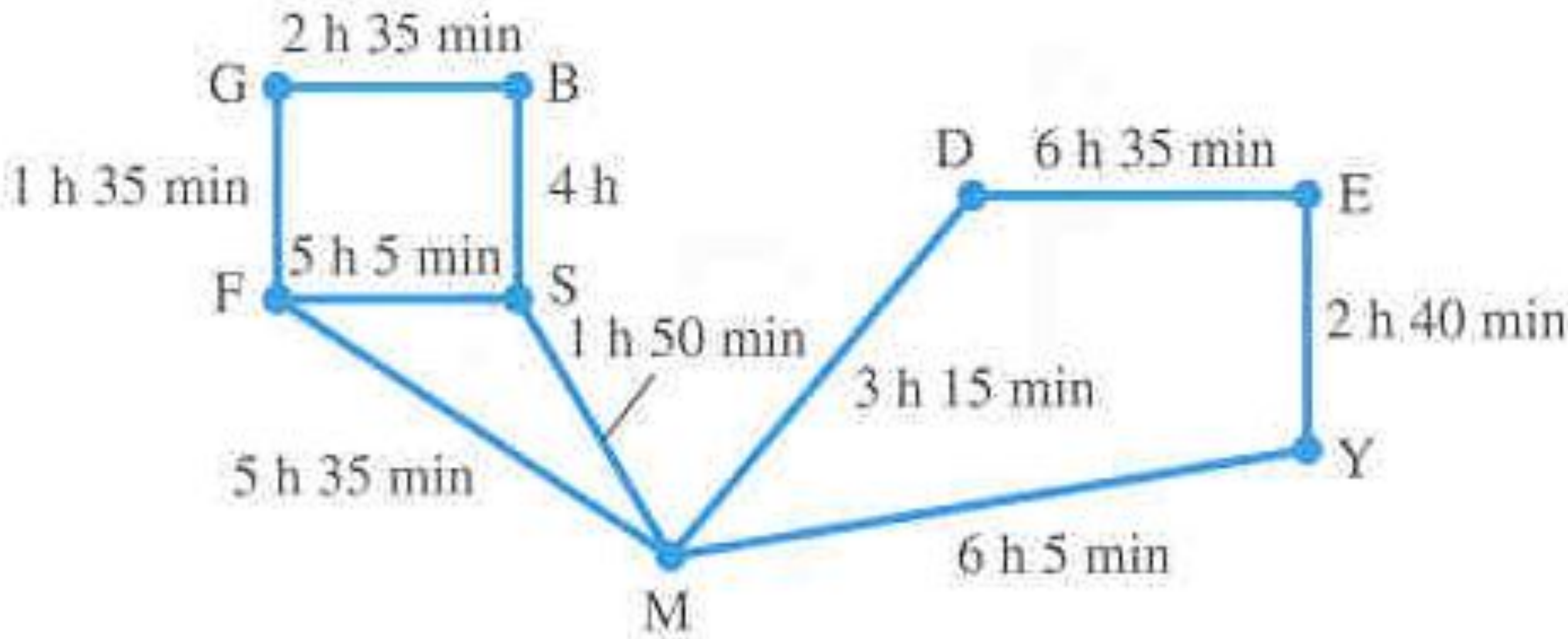


Use the given graph to answer the following:

1. Use vertices to describe two paths that start at vertex A (B) and end at vertex F.
2. Use vertices to describe a circuit that begins and ends at vertex F (G).
3. Use vertices to describe a path from vertex H to vertex E, passing through vertex I , but not through vertex G .
4. Use vertices to describe a path from vertex A to vertex I, passing through vertices B and G, as well as through the loop.
5. Explain why the sequences A, C, D, E, D and G, F, D, E, D are not paths.
6. Explain why the sequences A, C, D, G and H,I, F, E are not paths.

Walks & Paths & Circuits

When planning your trip around the Upper Midwest Region, it would be useful the length to know the length of time the bus ride takes. We might write the time for each bus ride on the graph as shown below



A graph with numbers on the edges as shown above is called a <u>weighted graph.</u> The numbers on the edges are called weights.

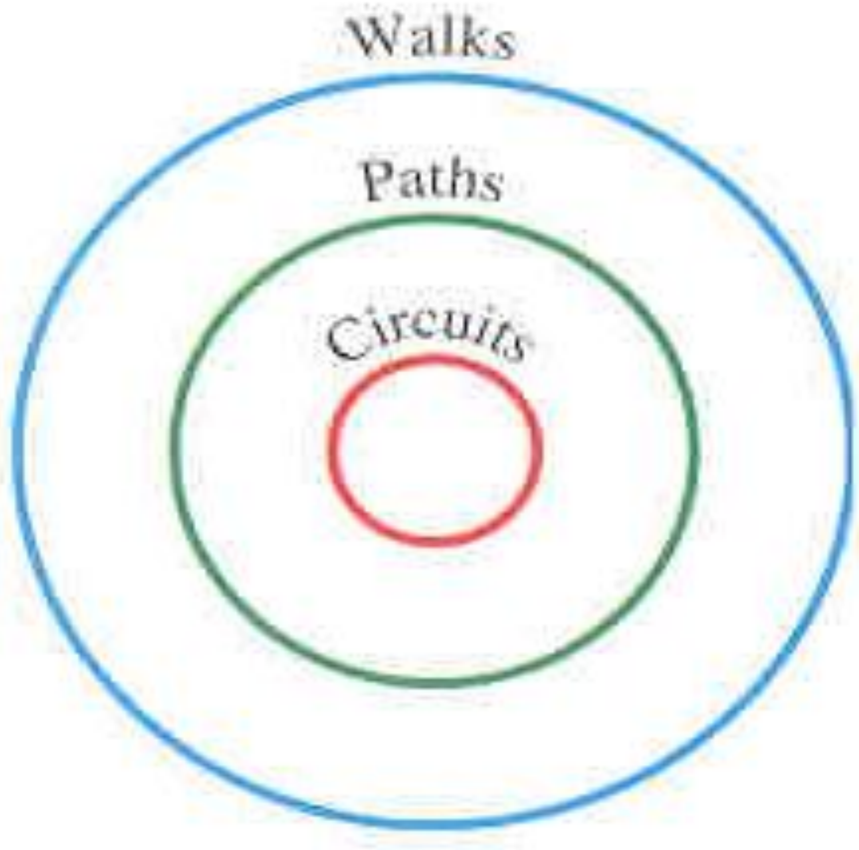
2. Graph Terminology

Walks & Paths & Circuits Summary

The relations among Walks, Paths and Circuits are shown below

A path of length *n* is a sequence of *n* edges that begins at a vertex of a graph and travels from vertex to vertex along edges of the graph.

The path is a circuit (or cycle) if it begins and ends at the same vertex.



Illustrative Example

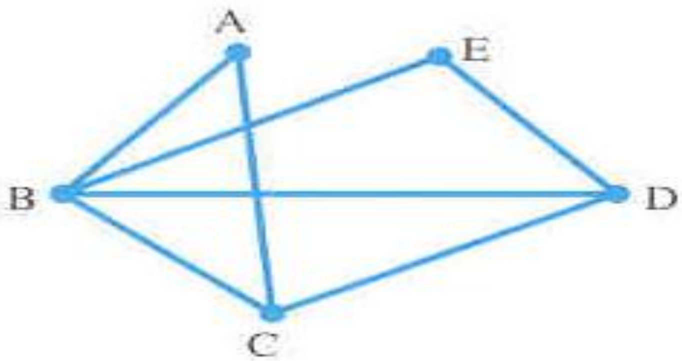
- Using the given graph classify each of the following sequences as a walk, a path, or a circuit.

(a) $E \to C \to D \to E$

(b) $A \to C \to D \to E \to B \to A$

(c) $B \to D \to E \to B \to C$

(d) $A \to B \to C \to D \to B \to A$



Walk	Path	Circuit
(a) No (no edge E to C)	No*	No*
(b) Yes	Yes	Yes
(c) Yes	Yes	No
(d) Yes	No (edge AB is used twice)	No**

* If a sequence of vertices is not a walk, it cannot possibly be either a path or a circuit.

** If a sequence of vertices is not a path, it cannot possibly be a circuit, since a circuit is defined as a special kind of path.

Check Point

Does each of the following lists of vertices form a path in the following graph? Which path are simple? Which are circuits? What are the lengths of those that are paths?

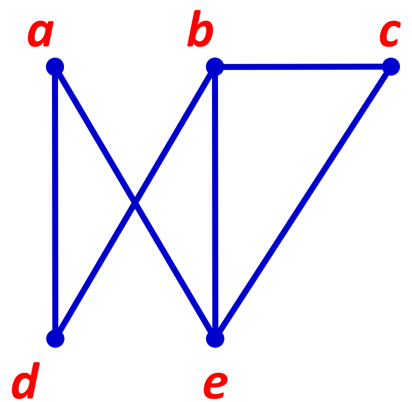


Figure 1

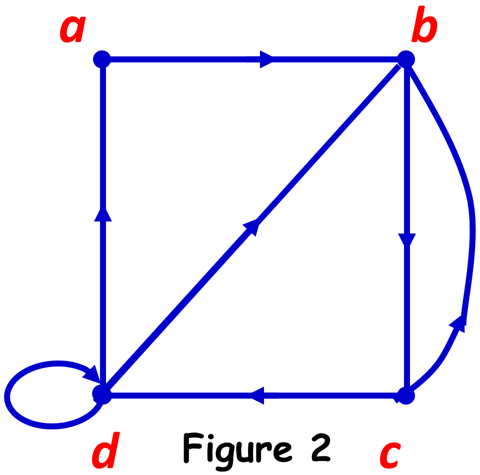


Figure 2

Figure 1

i) *a, e, b, c, b*

ii) *a, e, a, d, b, c, a*

iii) *e, b, a, d, b, e*

iv) *c, b, d, a, e, c*

Figure 2

i) *a, b, c, d, a*

ii) *a, d, d, b, c, b*

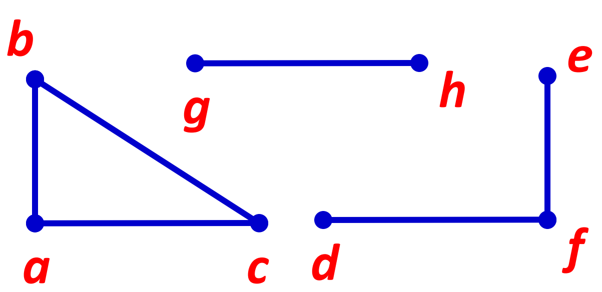
iii) *d, b, c, b*

iv) *c, a, b,* d

Connected Component

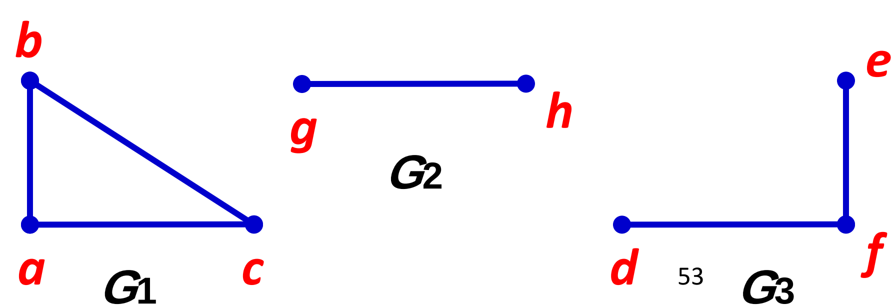
- A connected component of a graph *G* is a connected subgraph of *G* that is not a proper subgraph of another connected subgraph of *G*.
- A connected component of a graph G is a maximal connected subgraph of G.
- A graph *G* that is not connected has two or more connected components that are disjoint and have *G* as their union.

Example



Graph *G*

The graph *G* is the union of three disjoint connected subgraphs *G*1, *G*2, and *G*3. These subgraphs are the connected components of *G*.



Check Point

How many connected components does each of the following graphs have?

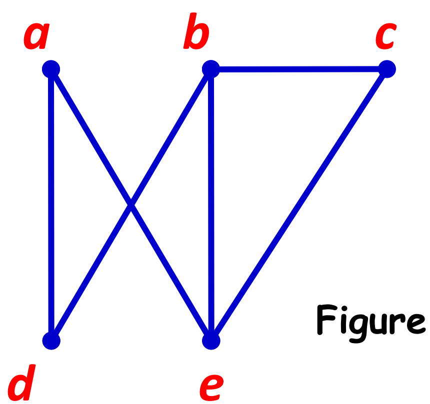
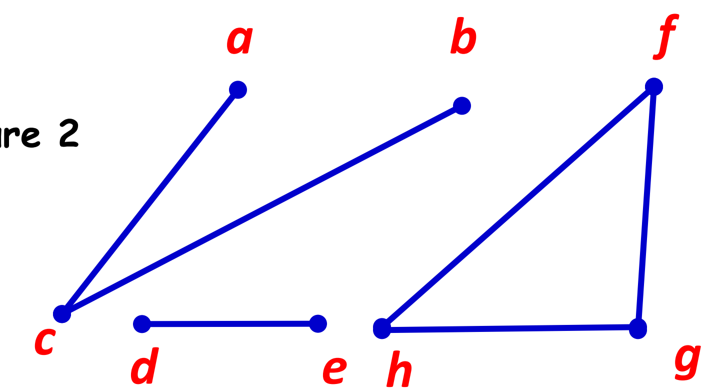


Figure 1

Figure 2



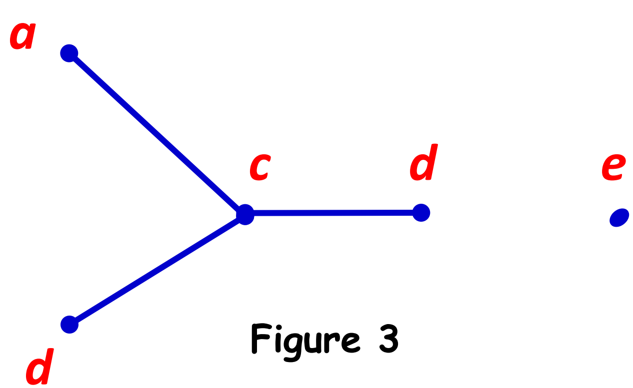


Figure 3

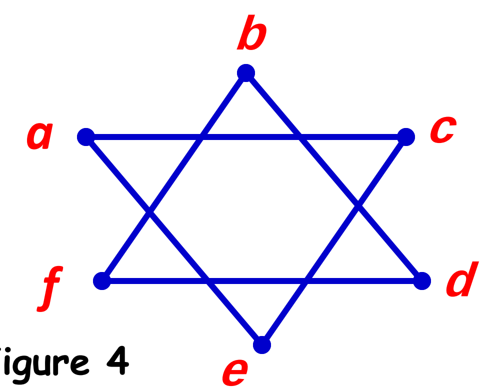


Figure 4

Connectedness الترابطية in Undirected Graphs

- An undirected graph is called connected if there is a path between every pair of distinct vertices of the graph.

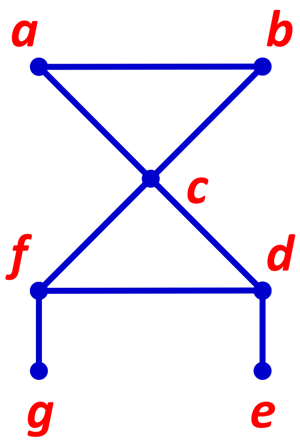


Figure 1

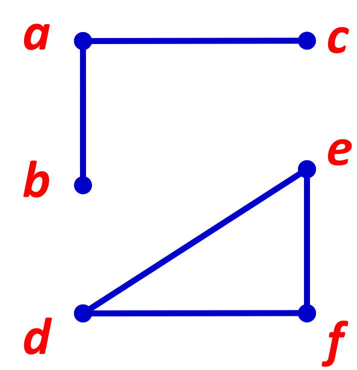


Figure 2

The graph is Connected since for every pair of distinct vertices there is path between them.

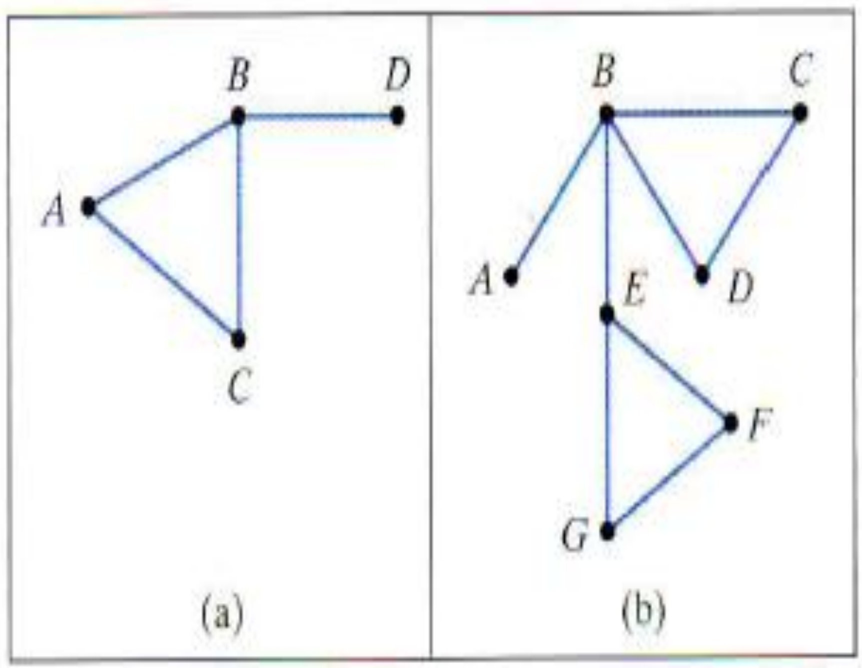
The graph is not connected since there is no path between vertices *a* and *d*.

Connected & Disconnected Graph

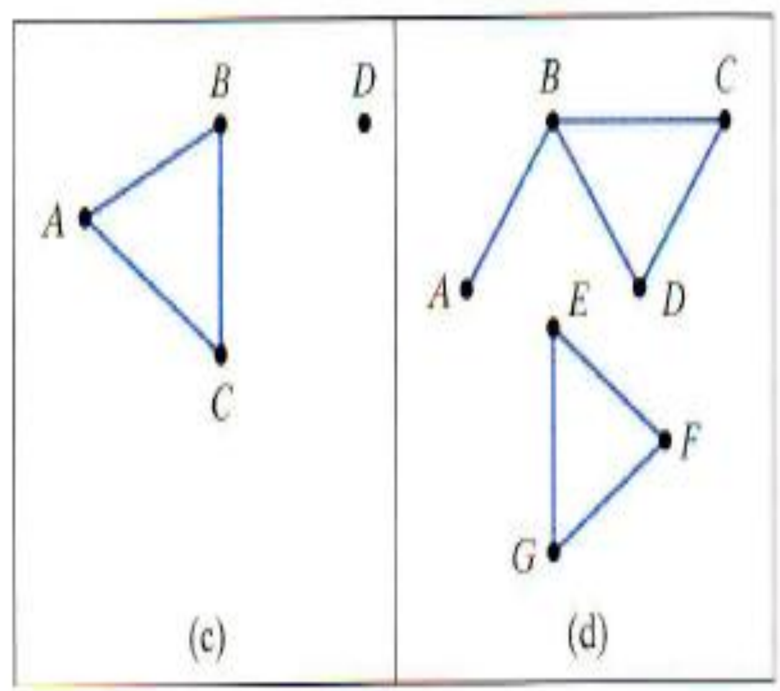
The words connected and disconnected are used to describe graphs. A graph is connected if for any two of its vertices there is at least one path connecting them.

Thus, a graph is connected if it consists of one piece. If a graph is not connected it is said to be disconnected. A disconnected graph is made up of pieces that are by themselves connected. Such pieces are called the components of the graph. See the figures below.

Connected Graphs



Disconnected Graphs



Check Point

Determine if each of the following is a connected graph.

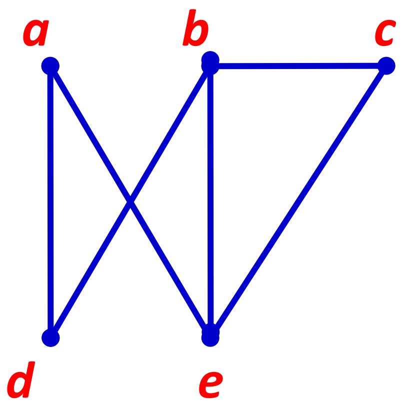


Figure 1

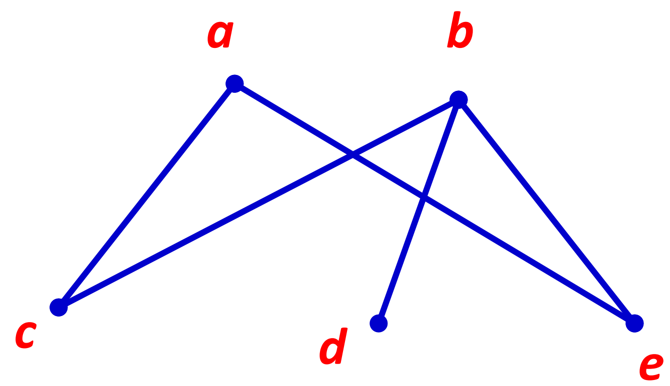


Figure 2

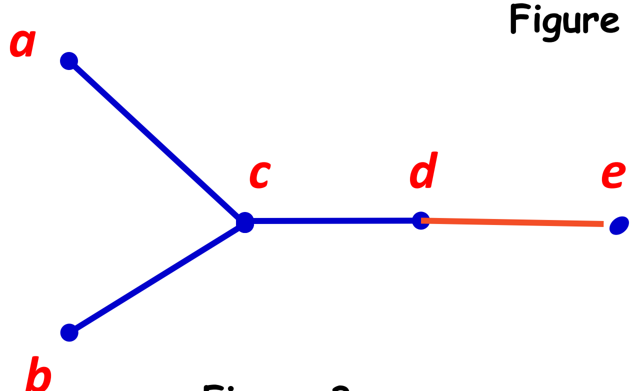
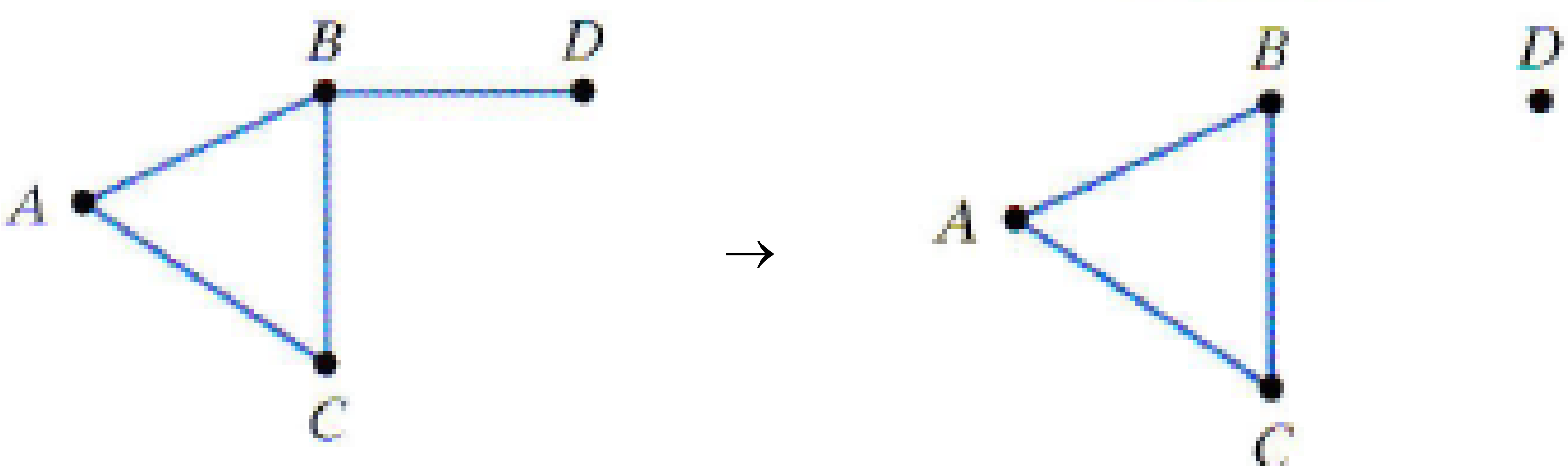


Figure 3

Cut/Bridge Edges

A cliche' آكلاشيه says that if you burn a bridge behind you, you'll never get back to where you were. This cliché tells us something about the word bridge in graph theory. A bridge is an edge that if removed from a connected graph would leave behind a disconnected graph.

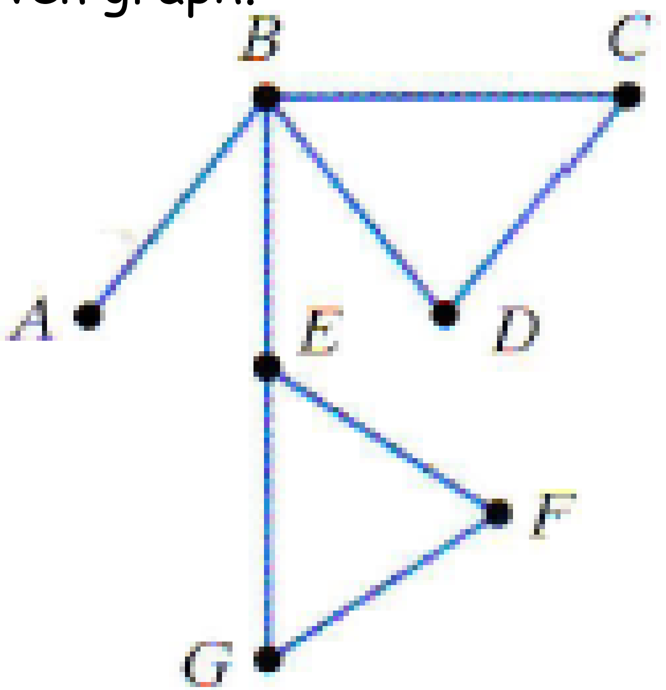
If edge BD were removed from Figure, shown below, vertex D would be isolated from the rest of the graph, leaving behind a disconnected graph. Thus BD is a bridge for the given graph.

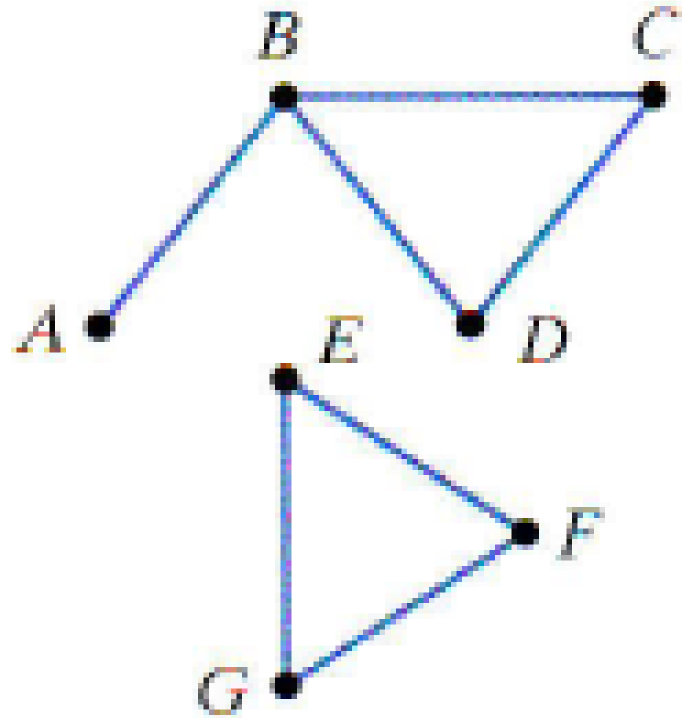


A cut edge in a graph is an edge whose removal disconnects a component of the graph

Cut/Bridge Edges

If edge BE were removed from the Figure shown below, the graph, leaving behind become a disconnected graph. As shown by the resulting two separated components. Thus BE is a bridge for the given graph.

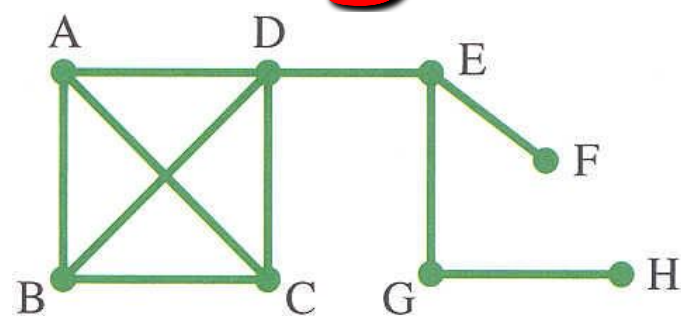




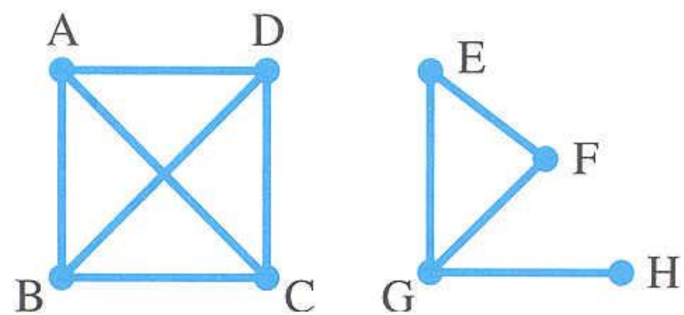
- An Edge is a cut edge (bridge) when the removal of this edge produces a subgraph that is not connected.

Cut/Bridge Edges

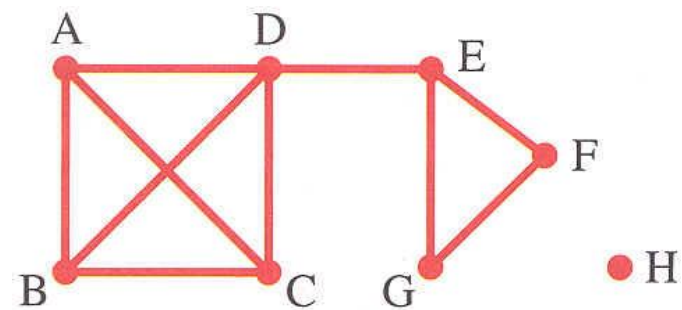
The graph in RHS has only one component, so we must look for edges whose removal would disconnect the graph.



DE is a cut edge; if we removed this edge, we would disconnect the graph into two components, obtaining graph (a) .



Also, HG is a cut edge; if we removed this edge, we would disconnect the graph into two components (one of which would be the single vertex H), obtaining graph (b)



None of the other edges is a cut edge; we could remove *anyone* of the other edges and still have a connected graph (that is, we would still have a path from each vertex of the graph to each other vertex). For example, if we remove edge GF, we end up with a graph that still is connected

Check Point

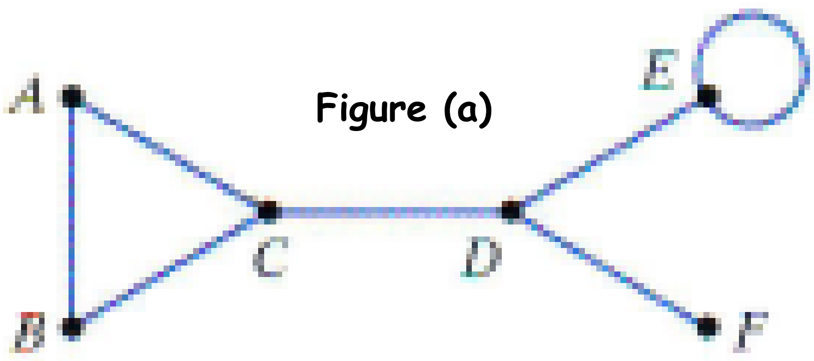


Figure (a)

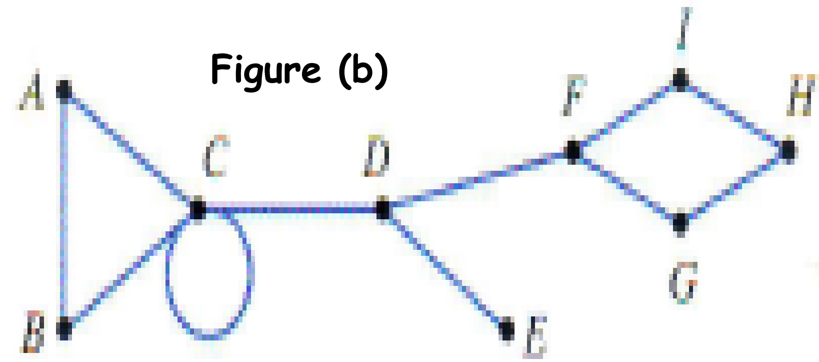


Figure (b)

<u>Use the graph (a) to answer the following:</u>

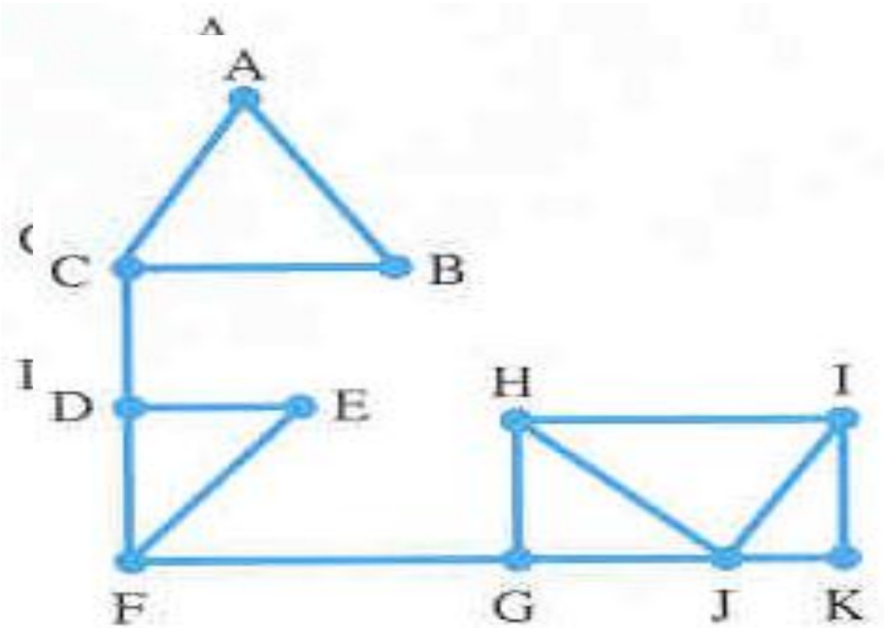
1. Explain why edges CD and DE are bridges
2. Identify an edge on the graph other than edges CD and DE that is a bridge.

<u>Use graph (b) to answer the following:</u>

1. Identify three edges that are bridges. Then show the components of the resulting graph once each bridge is removed.

Check Point

Identify all cut edges in the graphs 1-3. If there are none, say so.



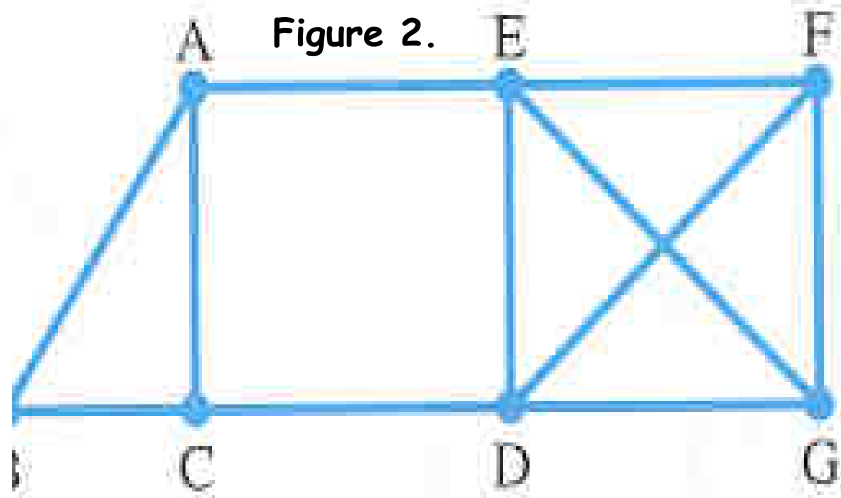


Figure 2.

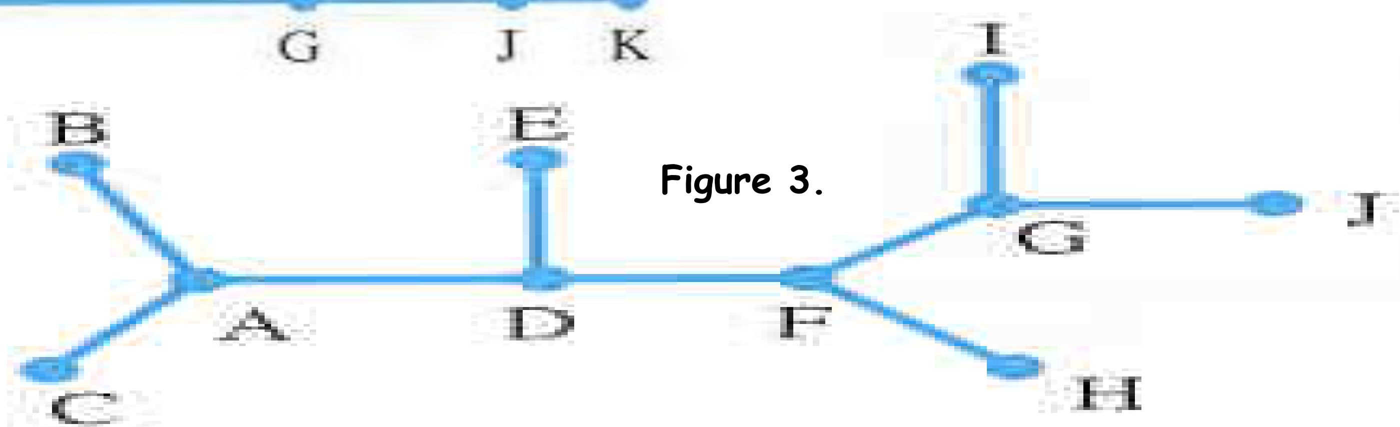


Figure 3.

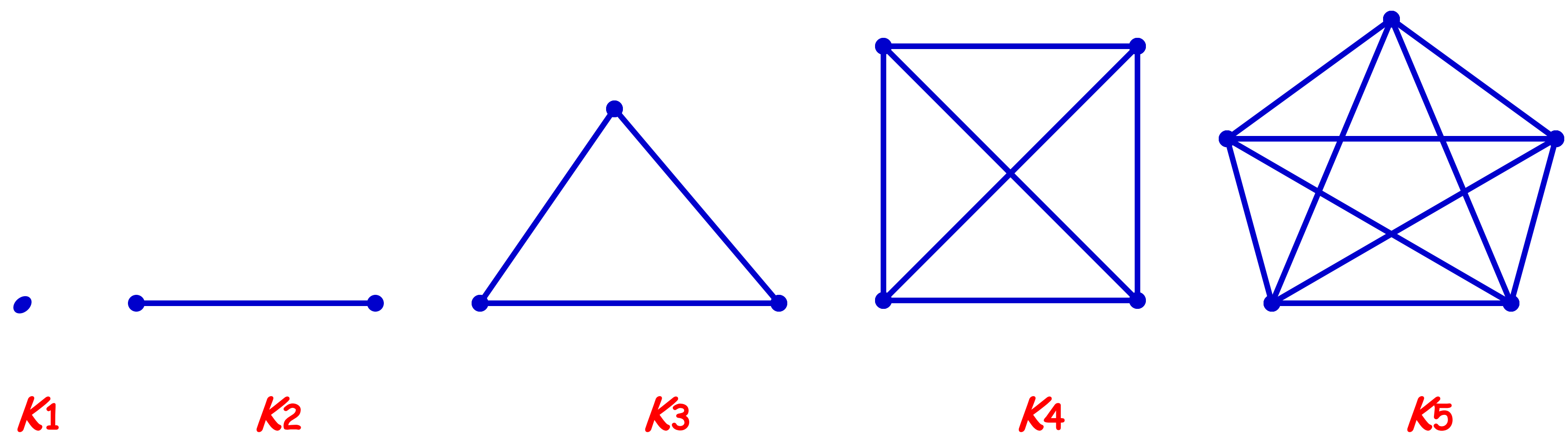
Some Special Classes of a Simple Graph

- Complete Graph
- Cycles
- Wheels

Complete Graph

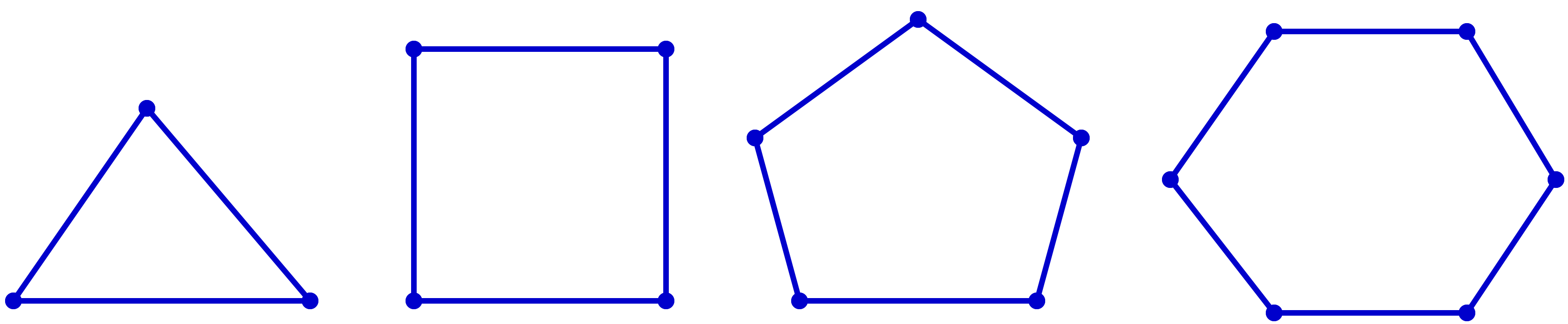
- The **Complete Graph** on n vertices, denoted by K_n, is the simple graph that contains exactly one edge between each pair of distinct vertices.

- **Example**: the graph K_n for $1 \leq n \leq 5$.



Cycles

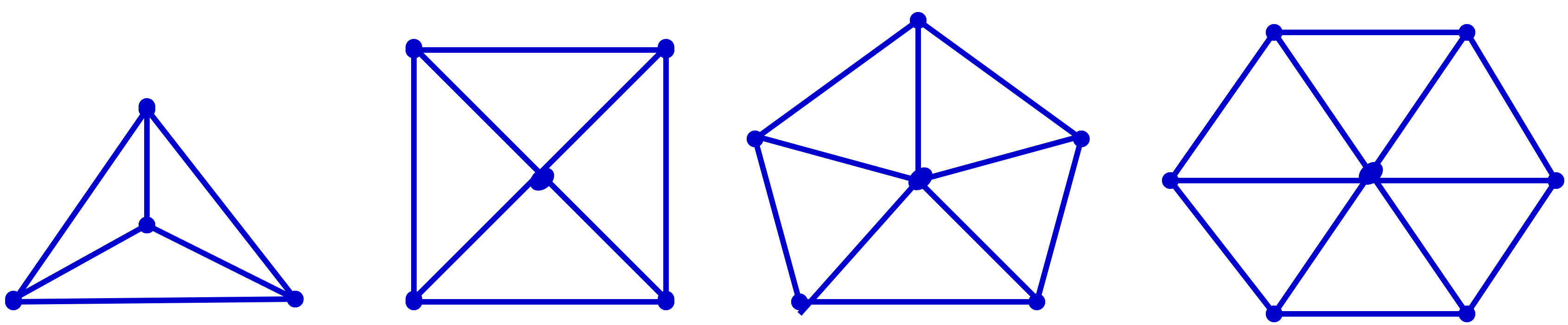
- The **Cycle** C_n, $n \geq 3$, consists of n vertices v_1, v_2, … v_n and edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, …, $\{v_{n-1}, v_n\}$ and $\{v_n, v_1\}$.
- **Example**: the C_n for $3 \leq n \leq 6$.



C_3 C_4 C_5 C_6

Wheels

- The **Wheel** W_n, $n \geq 3$, is obtain when we add an additional vertex (usually at the center) to the cycle C_n and connect this new vertex to each of the n vertices in C_n by new edges.
- **Example**: the W_n for $3 \leq n \leq 6$.

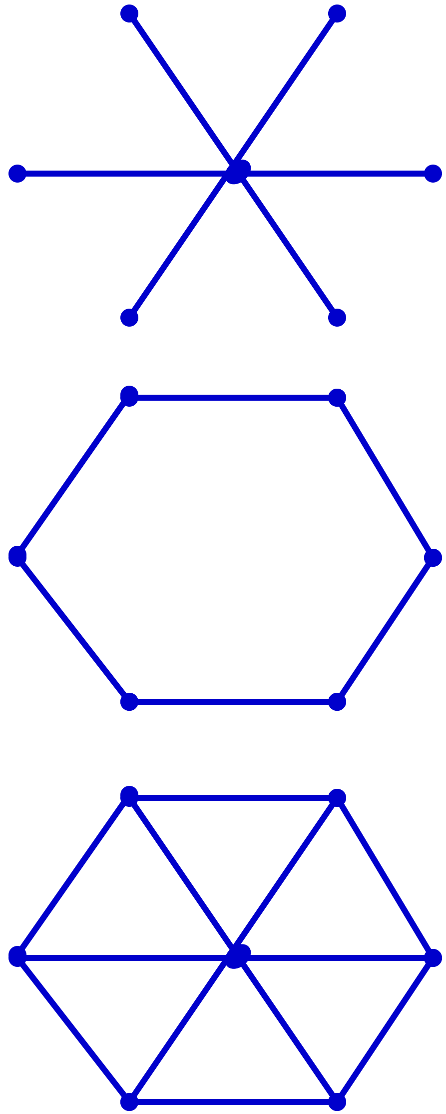


W_3 W_4 W_5 W_6

Applications of Special Types of Graph

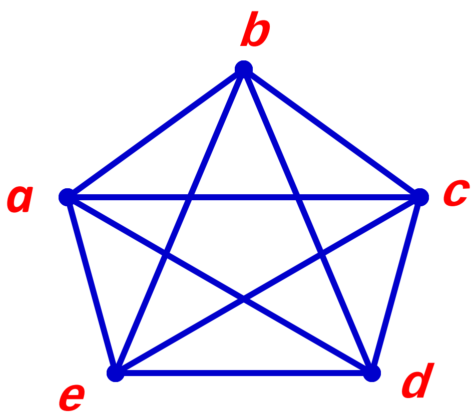
Local Area Network

- Illustrate the connection between various computers and devices
- Star topology : $K_{1,n}$
  - All devices connected to a central control device
- Ring topology : n Cycles (C_n)
  - Each device is connected to exactly two others
- Hybrid topology : star + ring
  - Redundancy makes the network more reliable

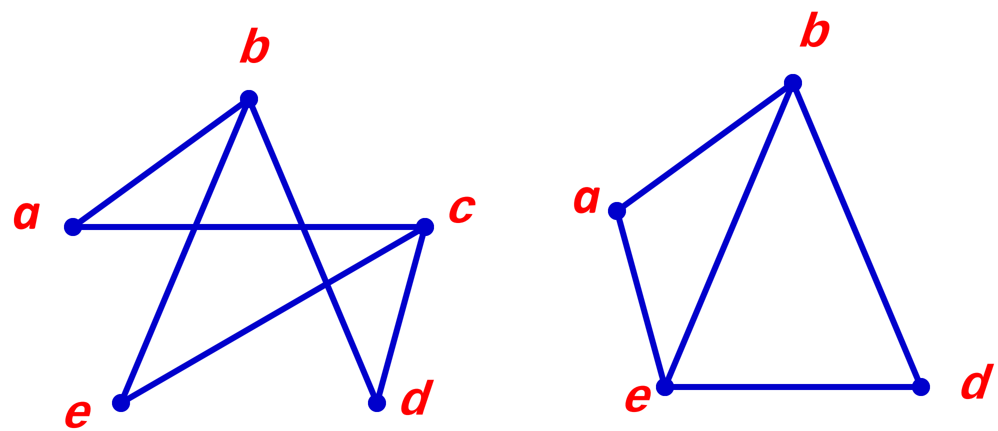


Subgraph

- A subgraph of a graph $G = (V, E)$ is a graph $H = (W, F)$, where $W \subseteq V$ and $F \subseteq E$.
- A subgraph H of G is a proper subgraph of G if $H \neq G$.



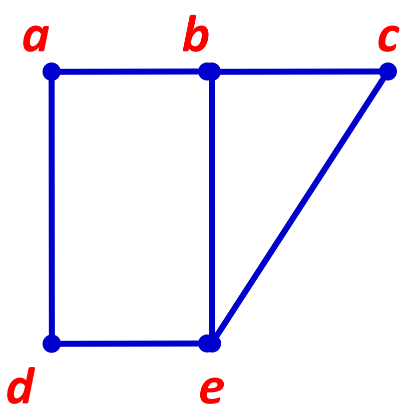
K_5



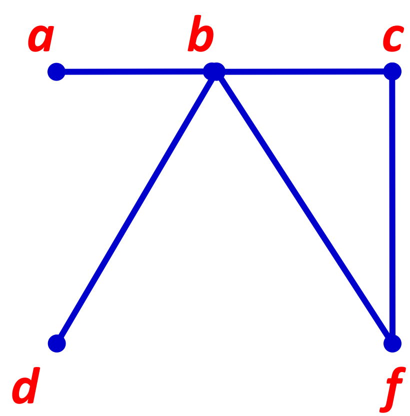
a few subgraph of K_5

Union of Graphs

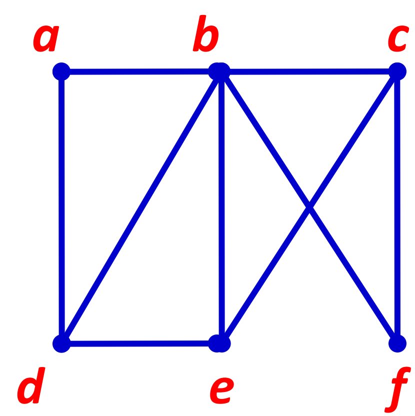
- The Union of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$.
- The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.



G_1



G_2

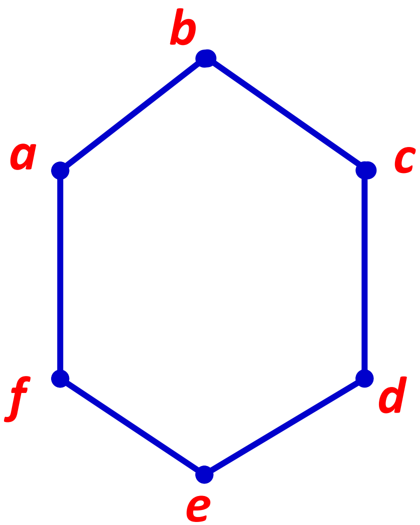


$G_1 \cup G_2$

Check Point

Find the union of the following graphs

Figure 1



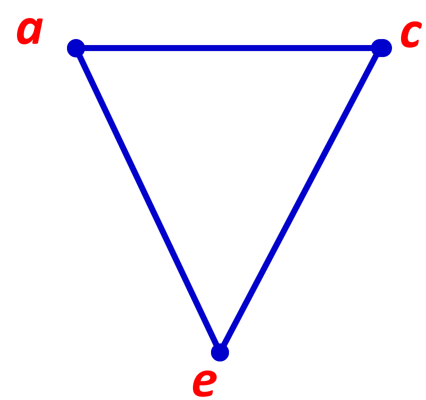
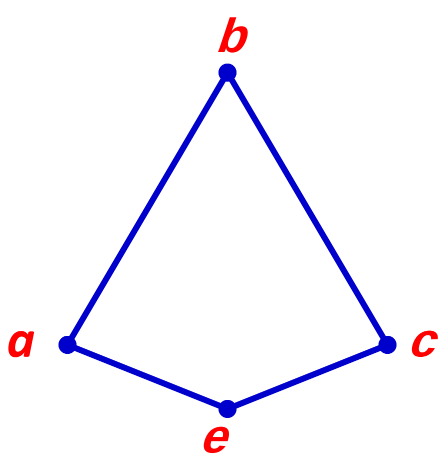


Figure 2

Figure 3



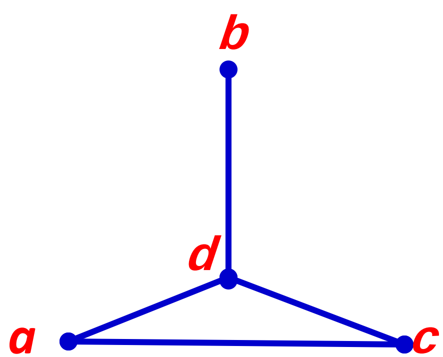
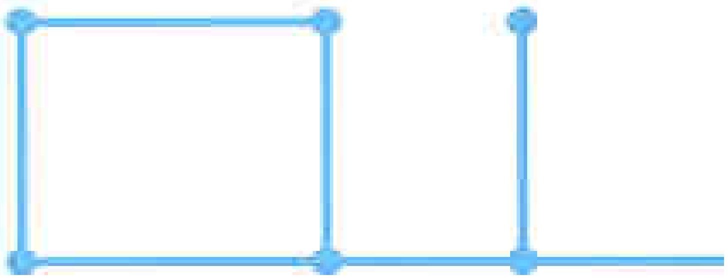


Figure 4

Exercise Set 2.

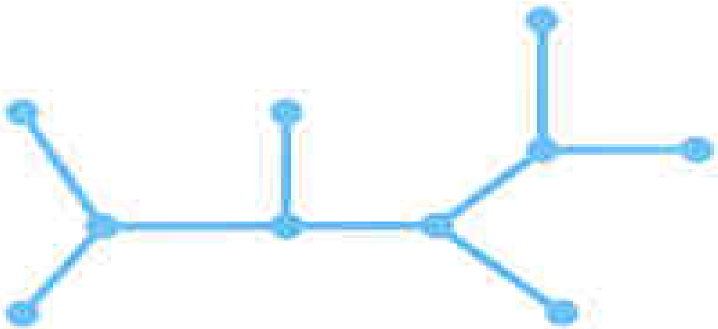
1. 6. For Exercises 1-6, determine how many vertices and how many edges each graph has

1.

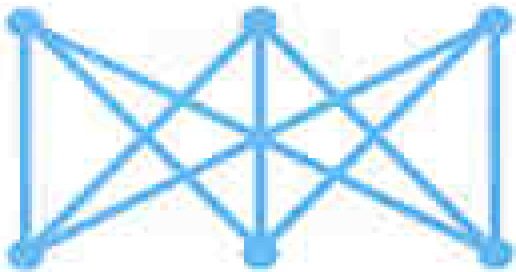


2.

3.



4.



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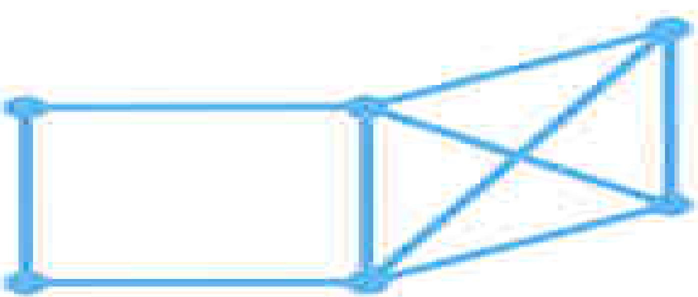
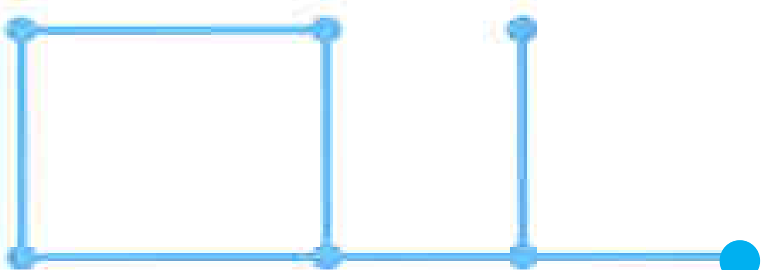


6.

Exercise Set 2.

7. – 10. For Exercises 7 -10, refer to the graphs shown in Exercises 1-4. For each of these graphs, find the degree of each vertex in the graph. Then add the degrees to get the sum of the degrees of the vertices of the graph. What relationship do you notice between the sum of degrees and the number of edges? Determine the cut edges if there are any?

7



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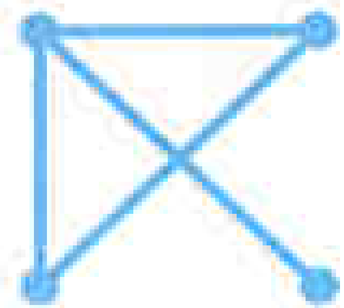

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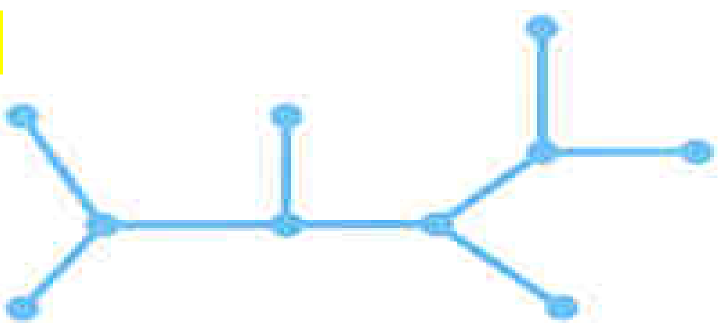

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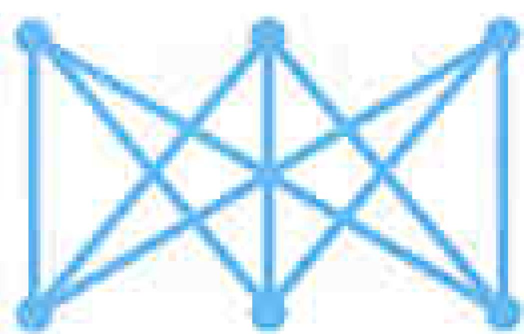
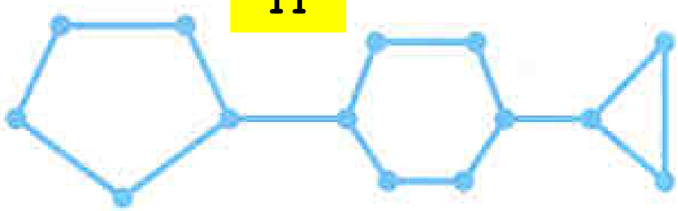


Exercise Set 2.

11. – 16. For Exercises 11-16, determine whether the graph is connected or disconnected. Then determine how many components the graph has.

11



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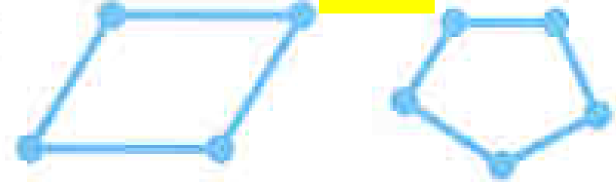

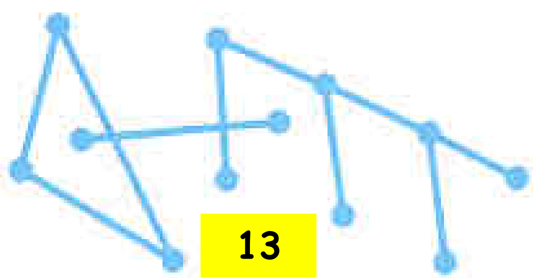

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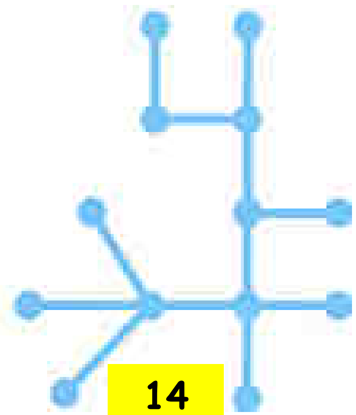

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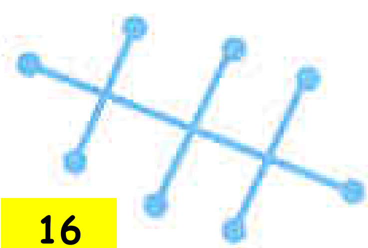


16

Exercise Set 2.

17. – 20. For Exercises 17 -20, use the theorem that relates the sum of degrees to the number of edges to determine the number of edges in the graph (without drawing the graph).

17. A graph with 4 vertices, each of degree 3.

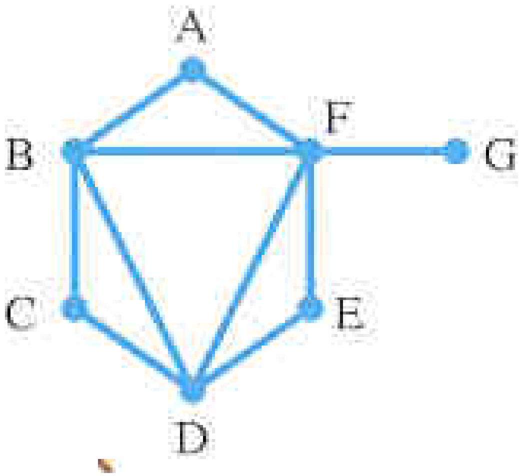
18. A graph with 8 vertices, each of degree 4.

19. A graph with 5 vertices, three of degree 1, one of degree 2, and one of degree 3.

20. A graph with 8 vertices, two of degree 1, three of degree 2, one of degree 3, one of degree 5, and one of degree 6.

Exercise Set 2.

21.-23. Referring to the given graph.



21. Which of the following are walks in the graph? If not, why not?

i. $D \to E$
ii. $B \to D \to F \to B \to D$
iii. $A \to B \to C$
iv. $E \to F \to A \to E$
v. $B \to A \to D$
vi. $C \to B \to C \to B$

22. Which of the following are paths in the graph? If not, why not?

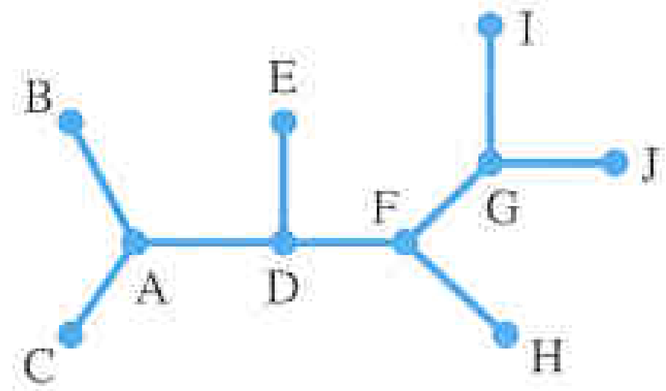
i. $B \to D \to E \to F$
ii. $D \to F \to B \to D$
iii. $B \to C \to D \to B \to A$
iv. $A \to B \to E \to F \to A$
v. $B \to D \to F \to B \to D$
vi. $D \to E \to F \to G \to F \to D$

23. Which of the following are circuits in the graph? If not, why not?

i. $C \to F \to E \to D \to C$
ii. $G \to F \to D \to E \to F$
iii. $A \to B \to C \to D \to E \to F$
iv. $A \to B \to D \to E \to F \to A$
v. $F \to D \to F \to E \to D \to F$

Exercise Set 2.

24. – 25. referring to the given graph.

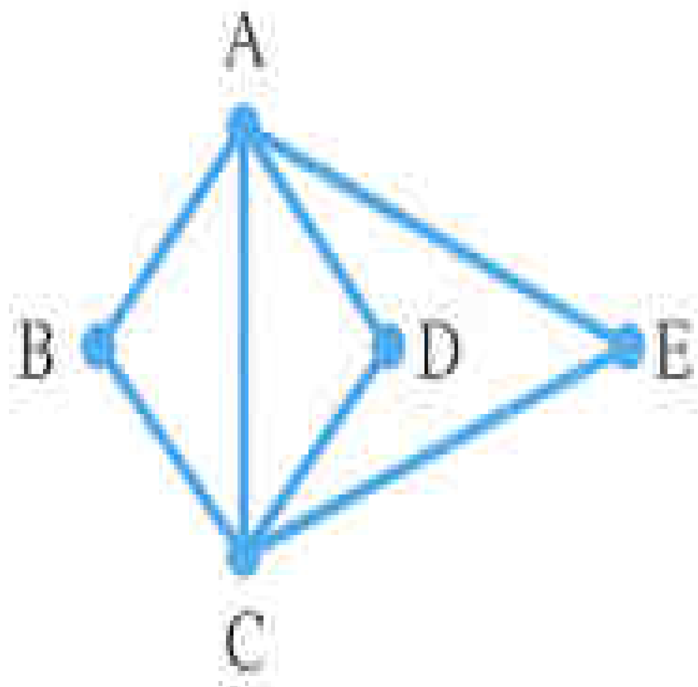


24. Which of the following are walks in the graph? If not, why not?
   i. $D \rightarrow F$
   ii. $I \rightarrow G \rightarrow J$
   iii. $I \rightarrow G \rightarrow J \rightarrow I$
   iv. $F \rightarrow G \rightarrow J \rightarrow H \rightarrow F$
   v. $B \rightarrow A \rightarrow D \rightarrow F \rightarrow H$
   vi. $B \rightarrow A \rightarrow D \rightarrow E \rightarrow D \rightarrow F \rightarrow H$

25. Which of the following are paths in the graph? If not, why not?
   i. $C \rightarrow A$
   ii. $A \rightarrow B \rightarrow C$
   iii. $C \rightarrow A \rightarrow D \rightarrow E$
   iv. $J \rightarrow G \rightarrow I \rightarrow G \rightarrow F$
   v. $D \rightarrow E \rightarrow I \rightarrow G \rightarrow F$
   vi. $C \rightarrow A \rightarrow D \rightarrow E \rightarrow D \rightarrow A \rightarrow B$

Exercise Set 2.

26. – 31. Exercises 26 - 31 refer to the following graph. In each case, determine whether the sequence of vertices is (i) a walk, (ii) a path, (iii) a circuit in the graph. .



26. $A \to B \to C$

27. $A \to B \to C \to D \to A$

28. $A \to B \to C \to D \to E$

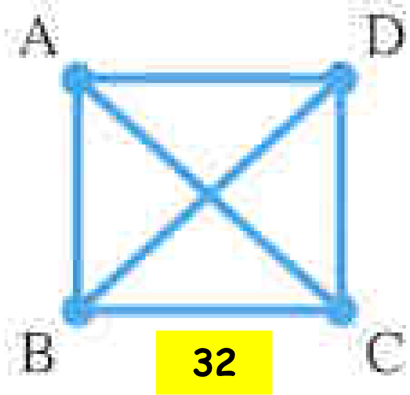
29. $A \to B \to A \to C \to D \to A$

30. $C \to A \to B \to C \to D \to A \to E$

31. $A \to B \to C \to A \to D \to C \to E \to A$

Exercise Set 2.

32. – 37. For each of Exercises 32-37, determine whether the graph is a complete graph. If not, explain why it is not complete.



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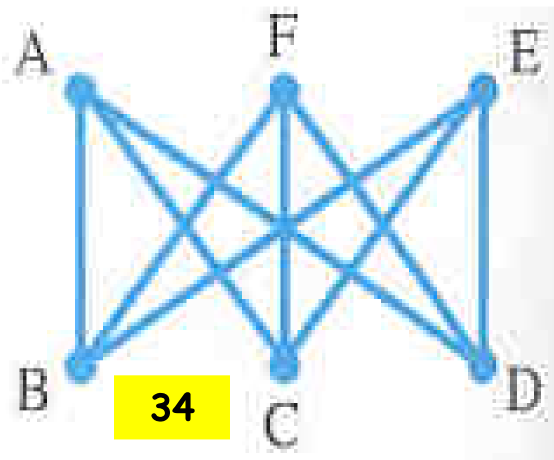

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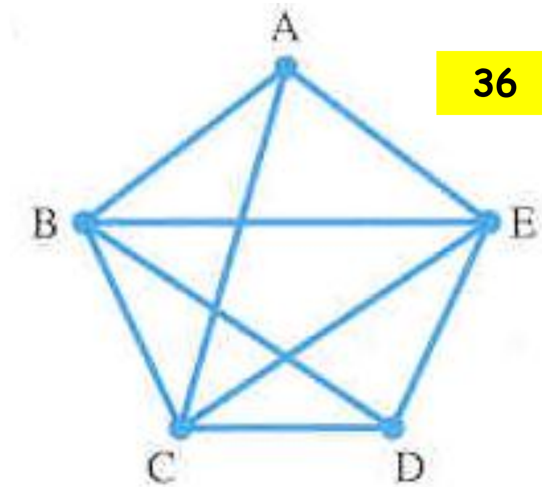

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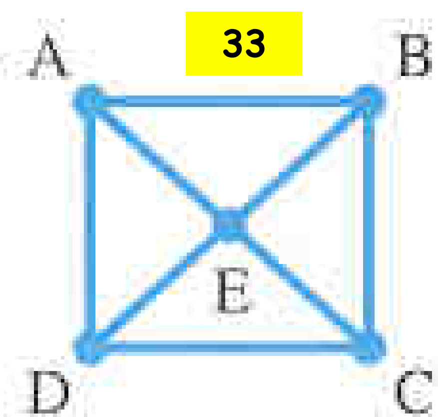

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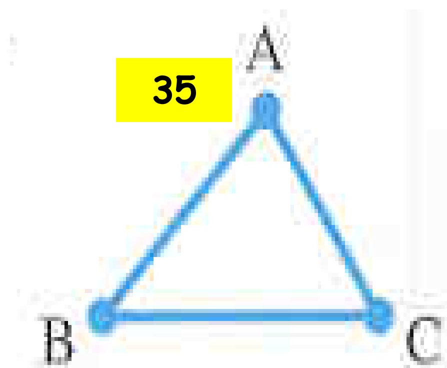

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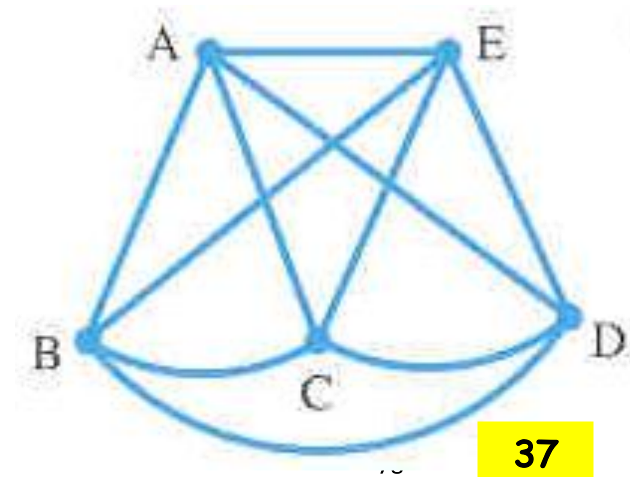


37

Exercise Set 2.

38. Students from two schools compete in chess. Each school has a team of four students. Each student must play one game against each student on the opposing team. Draw a graph with vertices representing the students, and edges representing the chess games. How many games must be played in the competition?

39. A chess master plays six simultaneous games with six other players. Draw a graph with vertices representing the players and edges representing the chess games. How many games are being played?

40. At a party there were four males- Tom, Joe, Chris, and Sam-and five females-Jennifer, Virginia, Mitzi, Karen, and Brenda. During the party each male danced with each female (and no female pair or male pair danced together). Draw a graph with vertices representing the people at the party and edges showing the relationship "danced with." How many edges are there in the graph?

Exercise Set 2.

41. A lawyer is preparing his argument in a libel case. He has evidence that a libelous rumor about his client was discussed in various telephone conversations among eight people. Two of the people involved had four telephone conversations in which the rumor was discussed, one person had three, four had two, and one had one such telephone conversation. How many telephone conversations were there in which the rumor was discussed among the eight people

42. Draw a graph with vertices representing the vertices (the corners) of a cube and edges representing the edges of the cube. In your graph, find a circuit that visits four different vertices. What figure does your circuit form on the actual cube?

43. Draw a graph with vertices representing the vertices (or corners) of a tetrahedron and edges representing the edges of the tetrahedron. In your graph, identify a circuit that visits three different vertices. What figure does your circuit form on the actual tetrahedron

Exercise Set 2.

41. Mary, Erin, Sue, Jane, Katy, and Brenda are friends at college. Mary, Erin, Sue, and Jane are in the same math class. Sue, Jane, and Katy take the same English composition class.
(a) Draw a graph with vertices representing the six students and edges representing the relation "take a common class."
(b) Is the graph connected or disconnected? How many components does the graph have?
(c) In your graph, identify a subgraph that is a complete graph with four vertices.
(d) In your graph, identify three different subgraphs that are complete graphs with three vertices. (There are several correct answers.)

42. Draw a graph whose vertices represent the faces of a cube and in which an edge between two vertices shows that the corresponding faces of the actual cube share a common boundary. What is the degree of each vertex in your graph? What does the degree of any vertex in your graph tell you about the actual cube?

3. Graph Representation And Graph Isomorphism

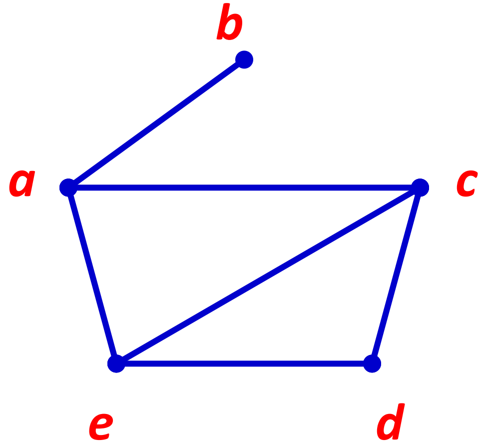
- Represent graphs using adjacency/ incidence lists
- Represent graphs using adjacency/ incidence matrix
- Represent graph using Representation matrix
- Isomorphic graphs

Representing Graphs On Computers

- Adjacency/Incidence List
  - Specify the vertices that are adjacent to each vertex of the graph
- Adjacency/Incidence Matrix
  - Represents graph by a matrix based on the adjacency of vertices
- Representation Matrix
  - Represents graph by a matrix based on the incidence of vertices and edges.

Adjacency/ Incidence List (Undirected Graph)

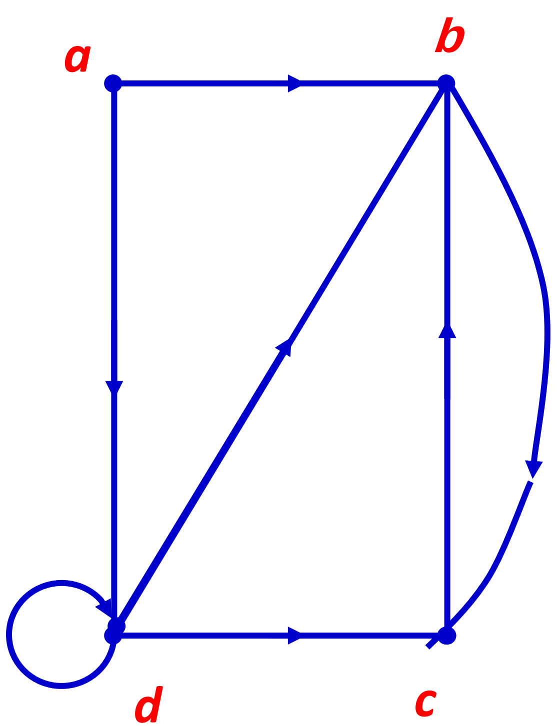
- Specify the vertices that are adjacent to each vertex of the graph



Adjacency List	
Vertex	Adjacent Vertices
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d

Adjacency/Incidence List (Directed Graph)

- List all the vertices that are the terminal vertices of edges starting at each vertex of the graph



Adjacency List	
Initial Vertex	Terminal Vertices
a	*b, d*
b	*c*
c	*b*
d	*b, c, d*

Check Point

Use an adjacency list to represent the given graph

Figure 1

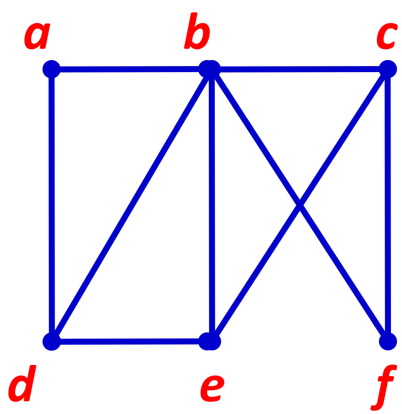


Figure 3

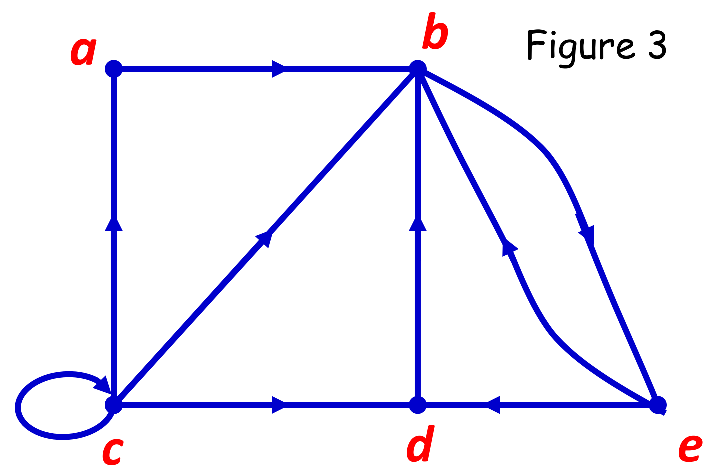


Figure 2

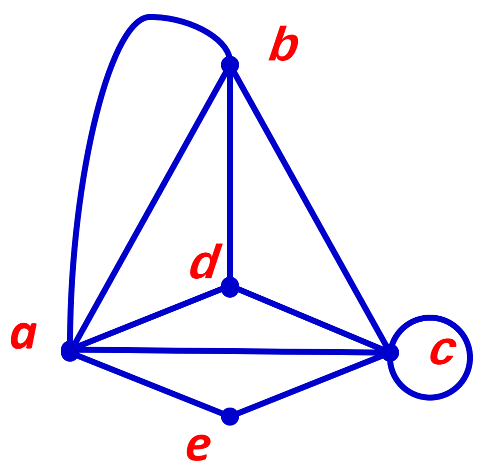
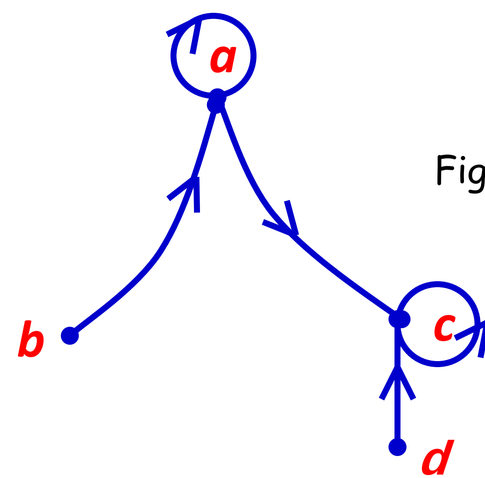


Figure 4



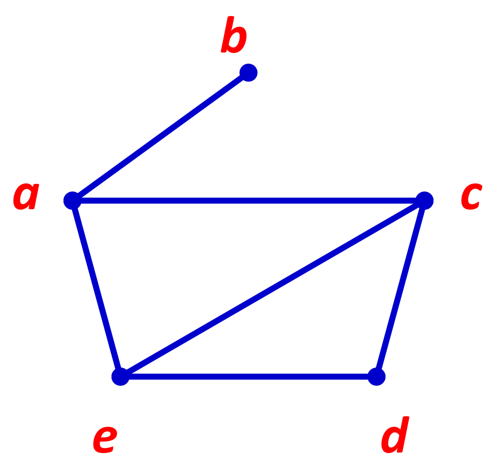
Adjacency/Incidence Matrix (Simple Undirected Graph)

- If **A** = [a_{ij}] is the adjacency matrix for simple undirected graph G with n vertices $v_1, v_2, \ldots, v_n$, then

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of } G \\ 0 & \text{otherwise} \end{cases}$$

A is symmetric

- Example 1: Use adjacency matrix to represent the graph below.

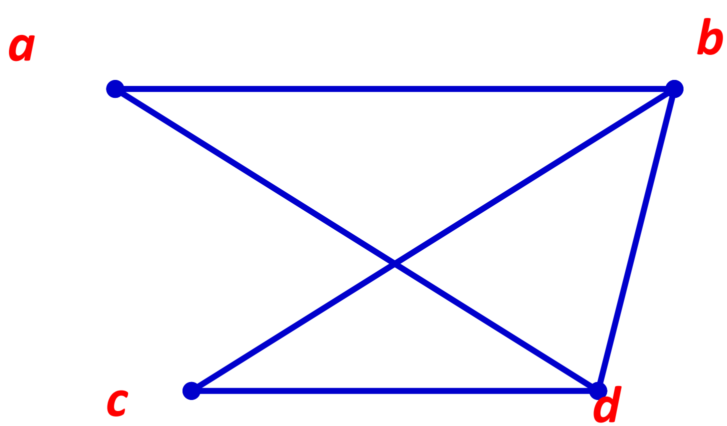


$$\mathbf{A} = \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}$$

Adjacency/Incidence Matrix (Simple Undirected Graph)

- Example 2: Use adjacency matrix to represent the graph below.

A is symmetric

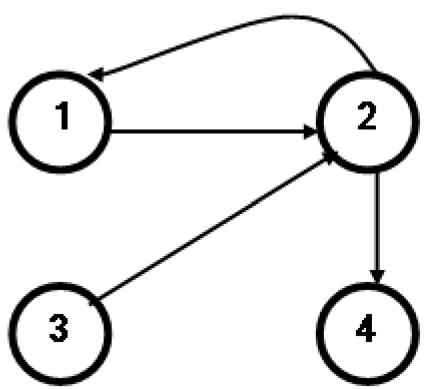


- $$\begin{array}{cc} \begin{array}{cccc} a & b & c & d \end{array} & \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} & \begin{array}{c} a \\ b \\ c \\ d \end{array} \end{array}$$

Adjacency/Incidence Matrix (Directed Graph)

- The incidence matrix for **undirected graph** G with n vertices $v_1, v_2, \ldots, v_n$, and m edges $e_1, e_2, \ldots, e_m$, is the $m \times n$ matrix
  - $\mathbf{M} = [m_{ij}]$, where
- $m_{ij} = 1$ if there is a directed edge from v_I to v_J and zero otherwise. This matrix need not be symmetric.

- Example 1: Use incidence matrix to represent the graph below.



To Vertex $\Rightarrow$

From Vertex $\Downarrow$

$$\begin{array}{cc} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Check Point

Represent each graph with an incidence matrix

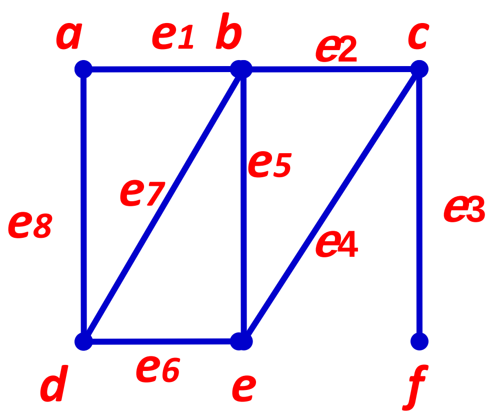


Figure 1

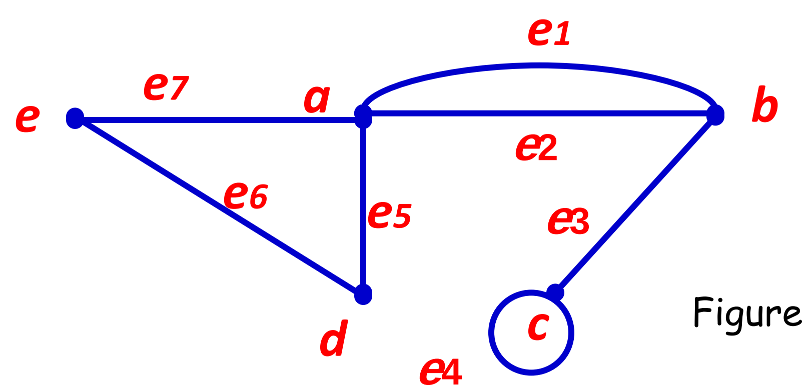


Figure 2

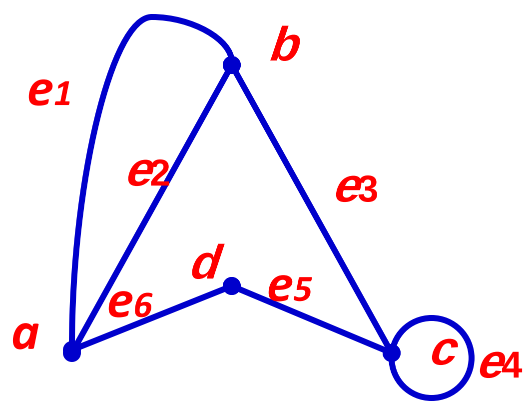
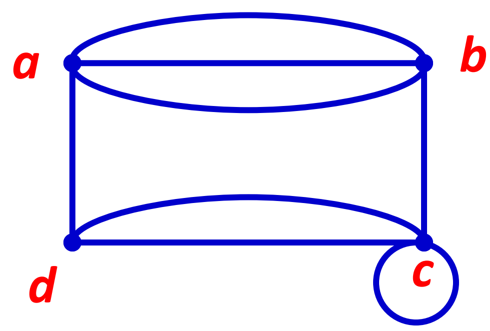


Figure 3

Representation Matrix (not Simple Undirected Graph)

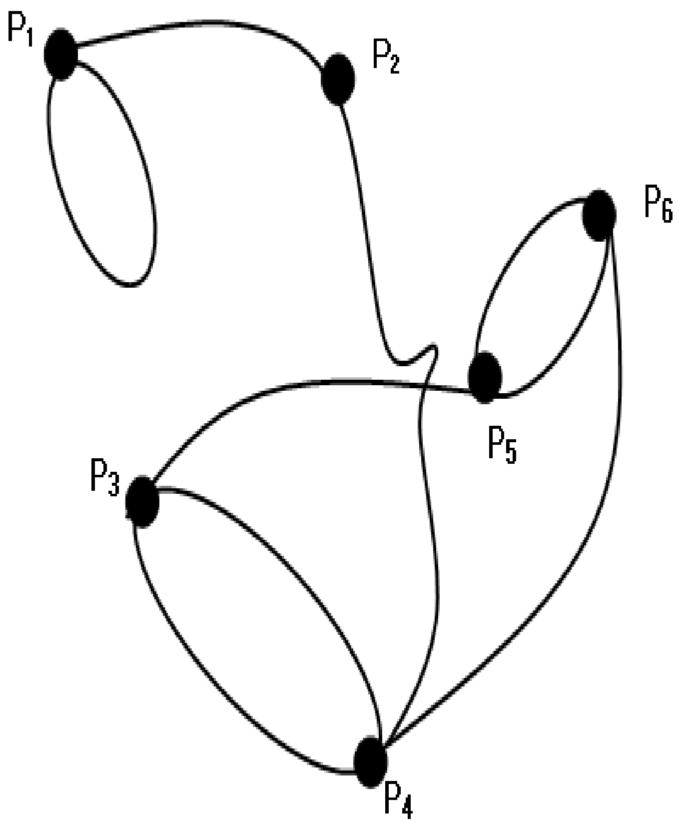
- A_{ij} is represent by the number of edges that associated to $[a_i, a_j]$
- A loop at a vertex represent by 1
- When multiple edges are present, adjacency matrix is no longer a zero-one matrix.
- The matrix is symmetric
- Example: Use Representation matrix to represent the graph below.



$$\mathbf{A} = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{cccc} a & b & c & d \end{array}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 3 & 0 & 1 \\ 3 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 2 & 0 \end{bmatrix} \begin{array}{c} a \\ b \\ c \\ d \end{array}$$

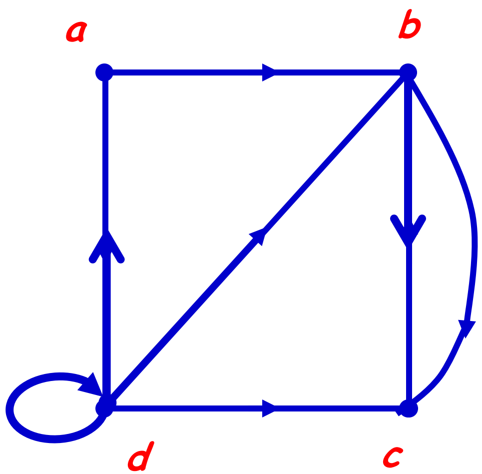
Representation Matrix (not Simple Undirected Graph)



$$
\begin{array}{c|c}
 & \begin{array}{cccccc} P_1 & P_2 & P_3 & P_4 & P_5 & P_6 \end{array} \\
\begin{array}{c} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{array} &
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 2 & 1 & 0 \\
0 & 1 & 2 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 2 \\
0 & 0 & 0 & 1 & 2 & 0
\end{bmatrix}
\end{array}
$$

Representation Matrix (Directed Graph)

- A_{ij} is represent by the number of edges from a_i to a_j
- A loop at a vertex represent by 1
- When multiple edges are present, adjacency matrix is no longer a zero-one matrix.
- The matrix is not symmetric.
- Example: Use adjacency matrix to represent the graph below.



$$\mathbf{A} = \begin{array}{c} \\ \\ \end{array}\begin{array}{cccc|c} a & b & c & d & \\ \end{array}$$

$$\begin{array}{c@{}c} \begin{array}{cccc} a & b & c & d \end{array} & \\ \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} & \begin{array}{c} a \\ b \\ c \\ d \end{array} \end{array}$$

Check Point

Represent the graph with a representation matrix

Figure 1

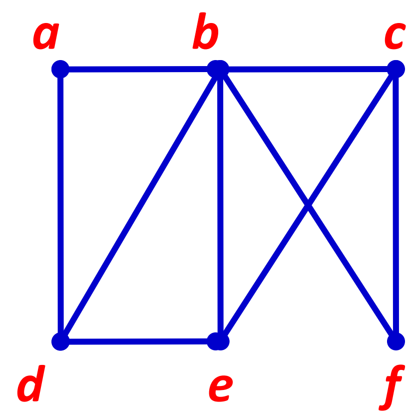


Figure 3

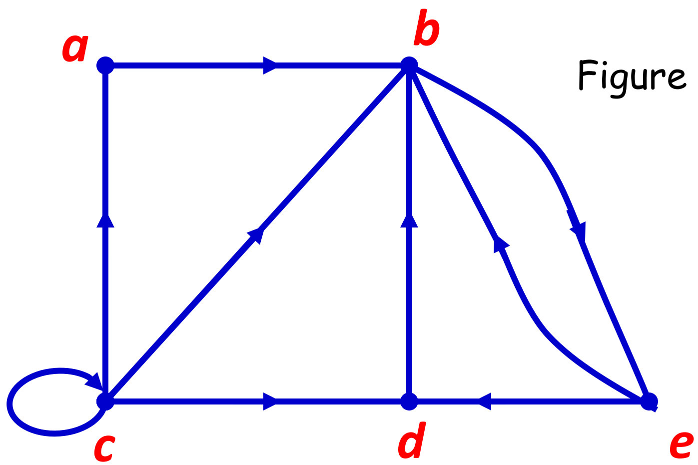


Figure 2

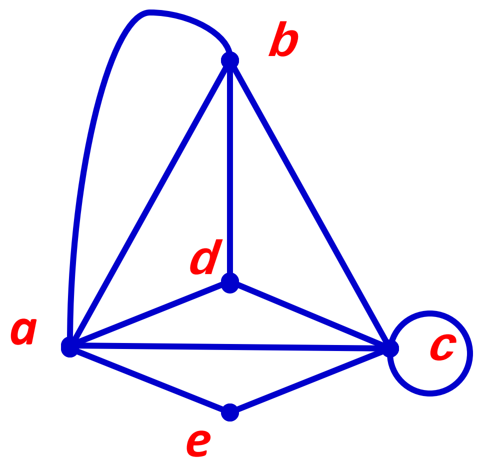
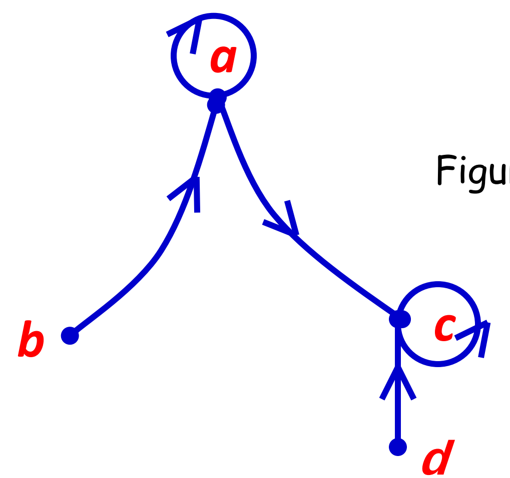


Figure 4

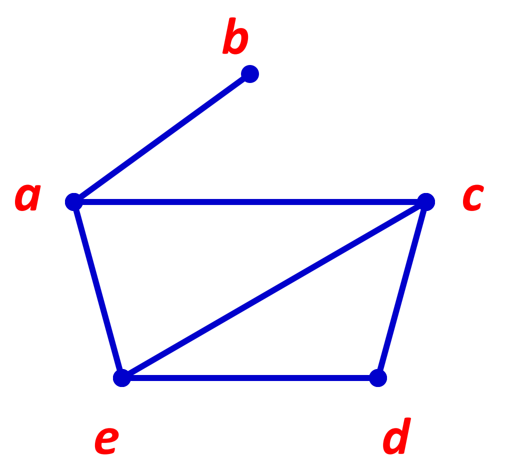


Drawing a Graph from Adjacency Matrix

- An adjacency matrix of a graph is based on the ordering chosen for the vertices.
- There are $n!$ different adjacency matrices for a graph with n vertices, because there are $n!$ different ordering of n vertices

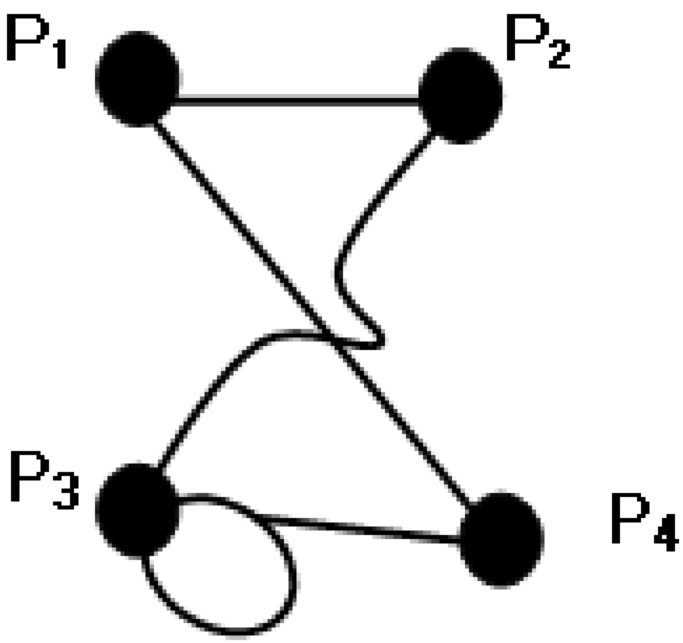
- <u>Example</u>: Draw a graph with the given adjacency matrix

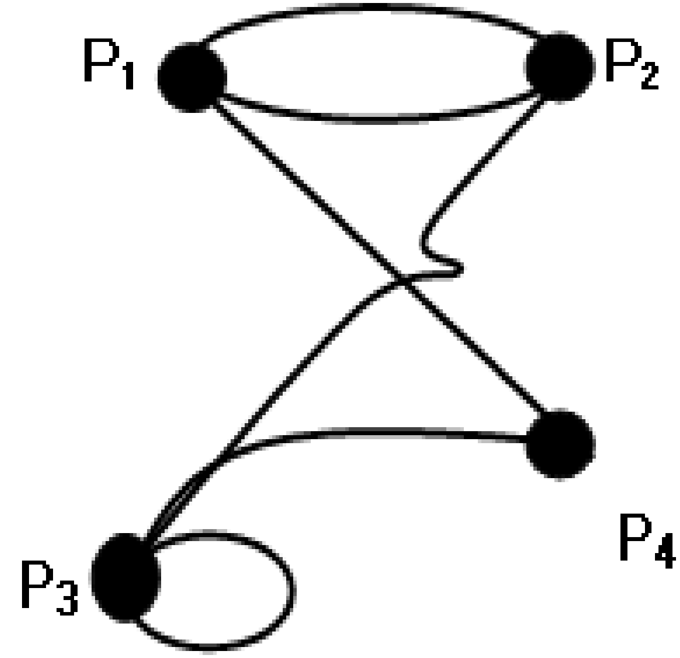
$$
\mathbf{A} = \begin{array}{c c} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ & \left[\begin{array}{ccccc} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{array}\right] \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} \end{array}
$$



Drawing a Graph from Adjacency Matrix

- The matrix $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$
- <u>Gives Rise To The Following Graphs</u>



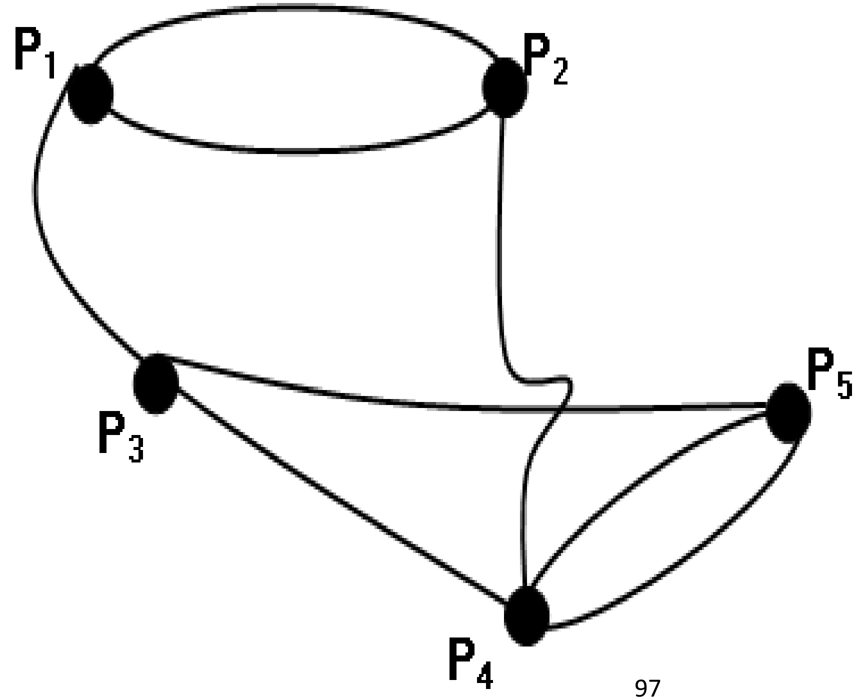


Drawing a Graph from Representation Matrix

- A graph can be drawn directly from its representation matrix as shown from the given example.

- Example: Draw a graph with the given adjacency matrix

$$\begin{bmatrix} 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 & 0 \end{bmatrix}$$



Check Point

Draw a graph with the given matrix

a)
$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Adjacency matrix

b)
$$\mathbf{A} = \begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

Representation matrix

c)
$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Representation matrix

Isomorphism of Graphs

- Two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a one-to-one correspondence between vertices of the two graphs that preserves the adjacency relationship.
- Isomorphism of simple graphs is an equivalence relation.
- There are $n!$ possible one-to-one correspondence between the vertex sets of two simple graphs with n vertices.
- The word Isomorphism comes from the Greek words *isos*, meaning "same," and *morph*, meaning "form."

Isomorphism of Graphs

• Equal Graphs

- Drawings of graphs are useful in explaining or illustrating specific situations . Here one should be aware that a graph may be sketched in various ways . In fact, a graph is determined by the vertices and edges joining the vertices, not by the particular appearance of the configuration. Thus two graphs "G" and "G`" are equal if they have the same number of vertices : "$P_1$, $P_2$, ....$P_n$" and if in each case they can be relabeled in such a way that the number of edges between $P_i$ and $P_j$ is the same in both "G" and "G`" .

Isomorphism of Graphs

- <u>Different Sketches Of The Same Graph</u>

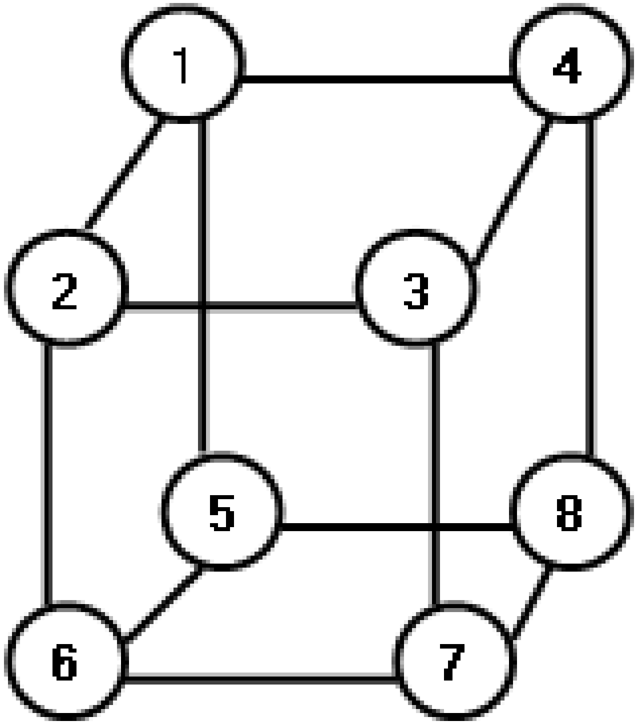
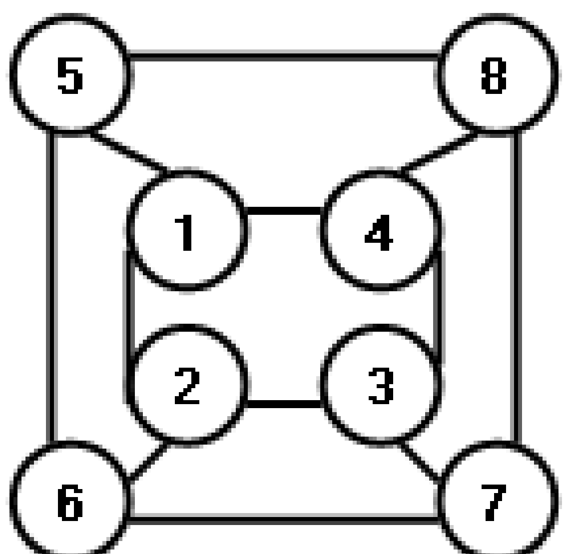


Figure 1

Figure 2



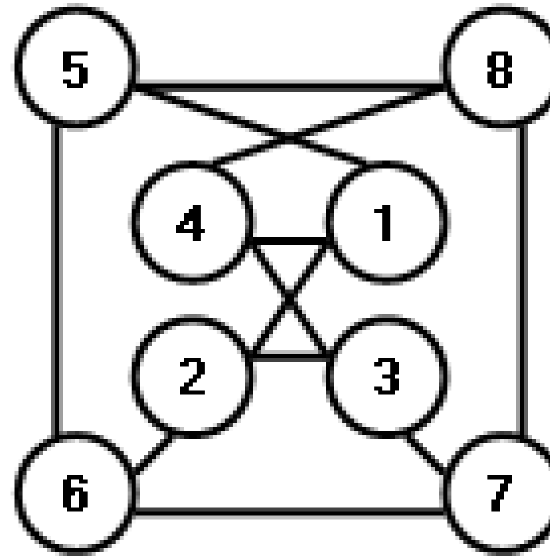


Figure 3

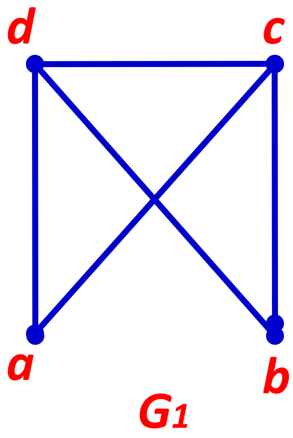
Isomorphism of Graphs (Conditions Tips)

- For Two **simple graphs** $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ to be **isomorphic** :
  - $|V_1| = |V_2|$
  - $|E_1| = |E_2|$
  - The corresponding vertices in G_1 and G_2, will have the same degree.

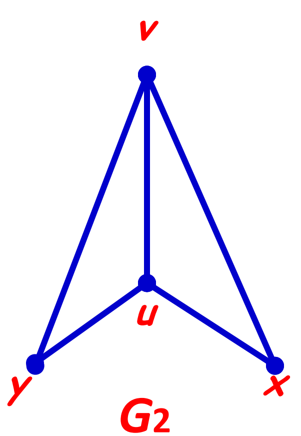
Hence:

Isomorphism is invariant

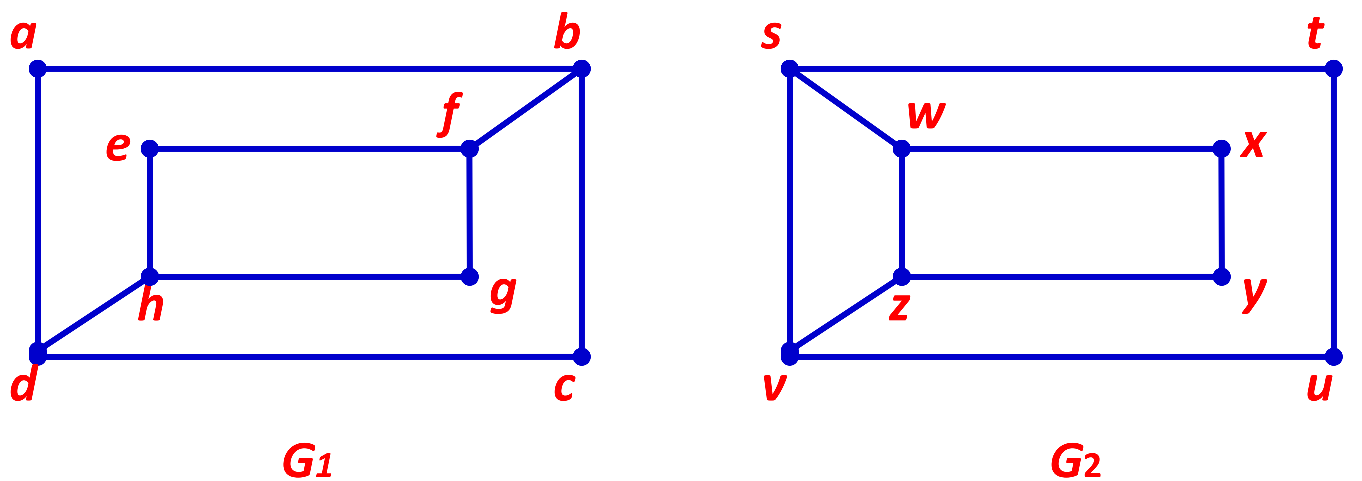
Example 1: Isomorphism of Graphs



- The graphs G_1 and G_2 are isomorphic since:
  - $|V_1| = |V_2| = 4$ and $|E_1| = |E_2| = 5$
  - $\deg(a) = \deg(x) = 2$,
  - $\deg(b) = \deg(y) = 2$,
  - $\deg(c) = \deg(v) = 3$,
  - $\deg(d) = \deg(u) = 3$.



Example 2: Isomorphism of Graphs



- The graphs G_1 and G_2 are not isomorphic since:
  - $|V_1| = |V_2| = 8$ and $|E_1| = |E_2| = 10$
  - **BUT**
  - deg $(a) = 2$, in G_1 .
  - a must correspond to either t, u, x, or y in G_2 since deg (t) = deg (u) = deg (x) = deg $(y) = 2$.
  - However, each of these four vertices is adjacent to another vertex of degree 2 which is not true for a. (a adjacent with to another vertex of degree 3)

Isomorphism of Graphs (By adjacency matrix)

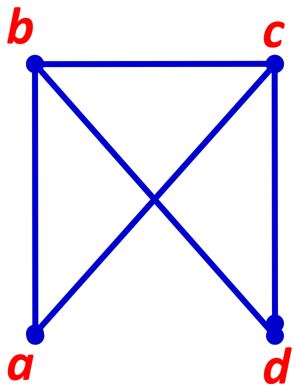
- For Two **simple graphs** $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ to be **isomorphic**

  – The adjacency matrix of G_1

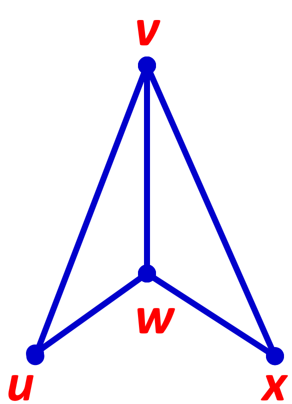
Is the same as

– The adjacency matrix of G_2, with rows and column are labeled by the images of corresponding vertices in G_1

Example 3: Isomorphism of Graphs



G_1



G_2

- The graphs G_1 and G_2 are isomorphic since:
  - $|V_1| = |V_2| = 4$

  And

  - $|E_1| = |E_2| = 5$
  - Adjacency matrix of G_1:

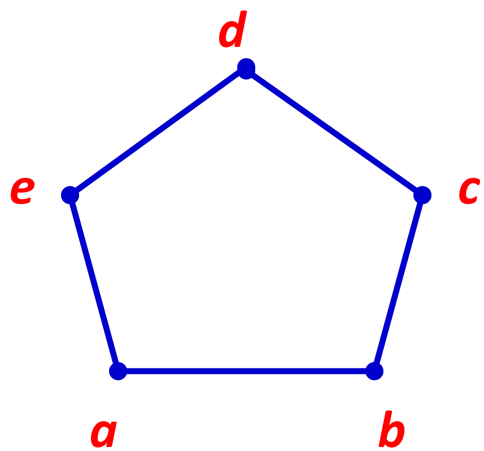
$$\begin{array}{cc} & \begin{array}{cccc} a & b & c & d \end{array} \\ \mathbf{A}_{G1} = & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{array}{c} a \\ b \\ c \\ d \end{array} \end{array}$$

  - Adjacency matrix of G_2 with rows and column are labeled by the images of corresponding vertices in G_1:

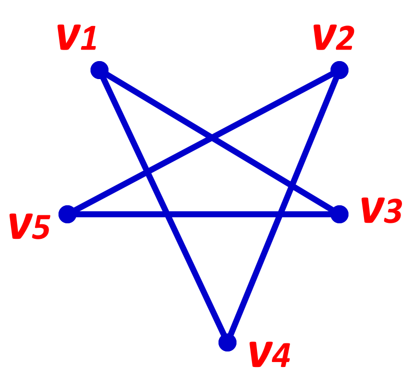
$$\begin{array}{cc} & \begin{array}{cccc} u & v & w & x \end{array} \\ \mathbf{A}_{G2} = & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{array}{c} u \\ v \\ w \\ x \end{array} \end{array}$$

Check Point

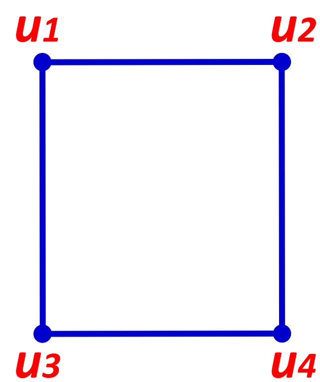
Determine if the following graph are isomorphic.



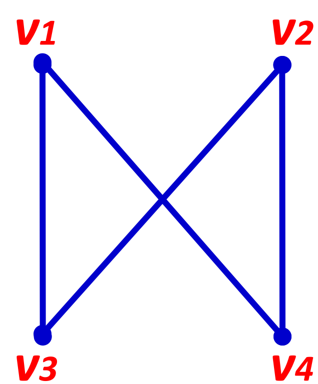
G1



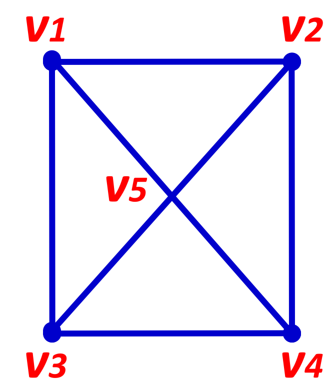
G2



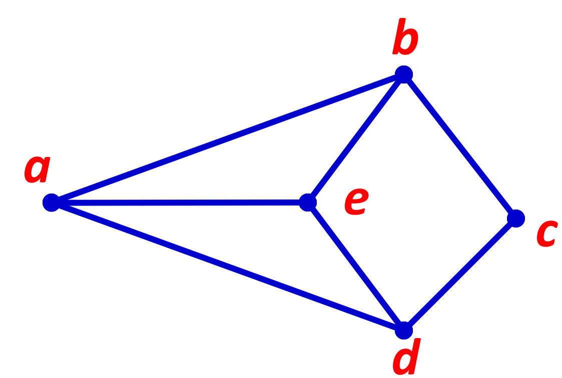
G



H



G1

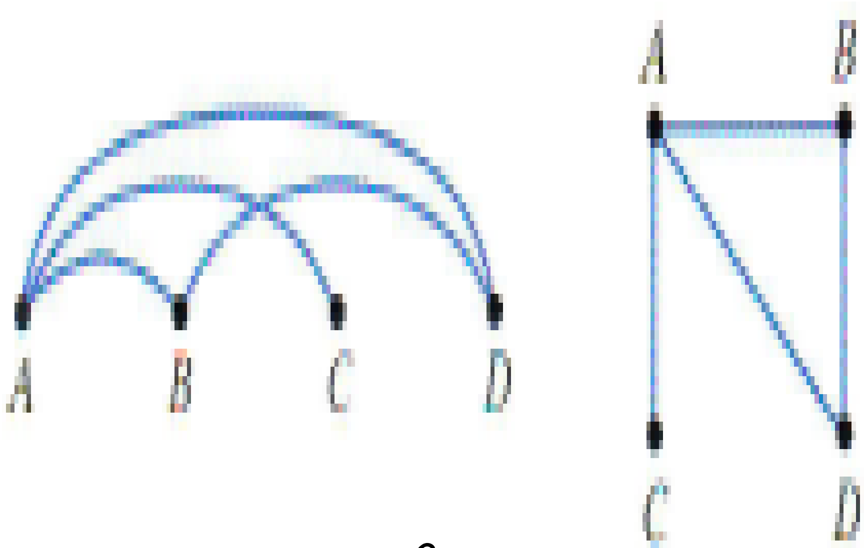


G2

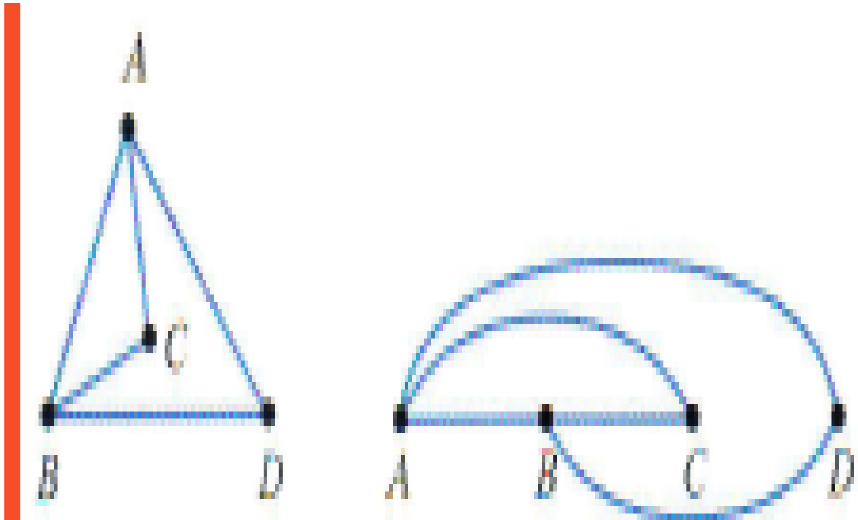
Exercise Set 3.

1. Draw two equivalent graphs for each description.
i. The vertices are A, B, C, and D. Three edges are AB, BC, BD, CD, and CC.
ii. The vertices are A, B, C, and D. The edges are AD, BC, DC, BB, and DB.

2. In Exercises 2.a and 2.b, explain why the two figures show equivalent graphs. Then draw a third equivalent graph.



2.a

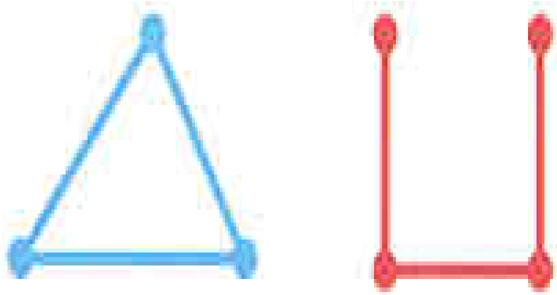


2.b

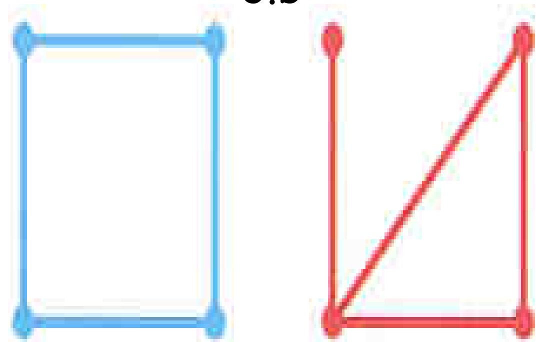
Exercise Set 3.

3. For Exercises 3.1 – 3.6, determine whether the two graphs are isomorphic. if so, label corresponding vertices of the two graphs with the same letters and color-code corresponding edges. (Note that there is more than one correct answer for many of these exercises

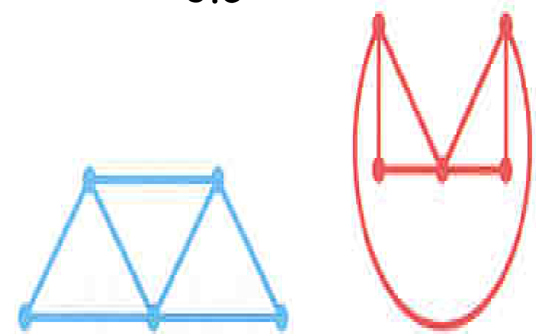
3.a



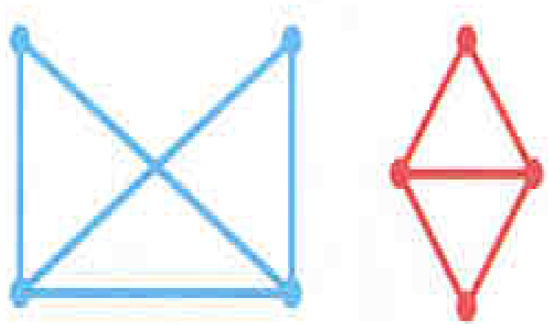
3.b



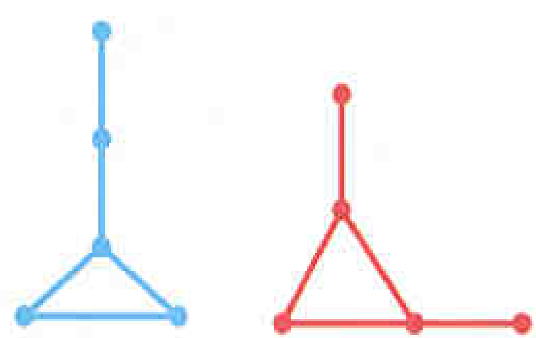
3.c



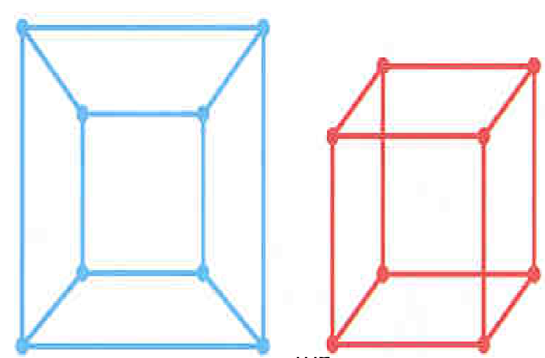
3.d



3.e



3.f

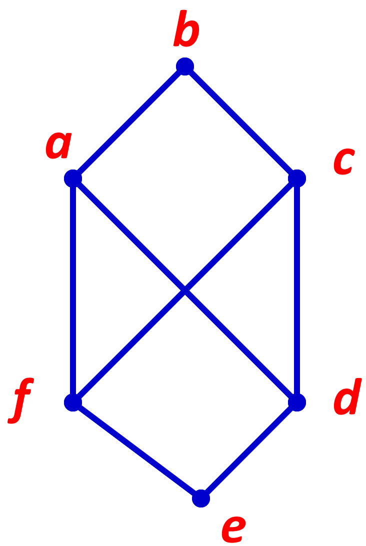


Paths and Isomorphism

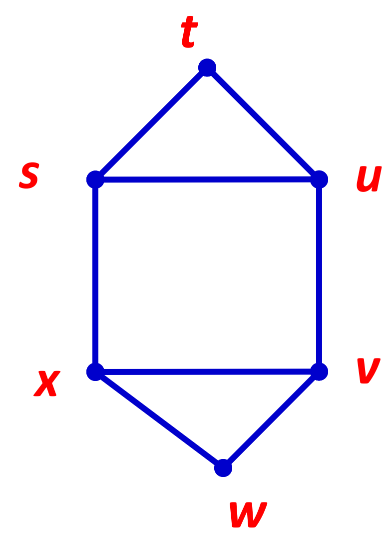
- Graphs, G_1 and G_2 are isomorphism only if in both graph exist a simple circuit of same length k (k >2).

Example

Graph *G*



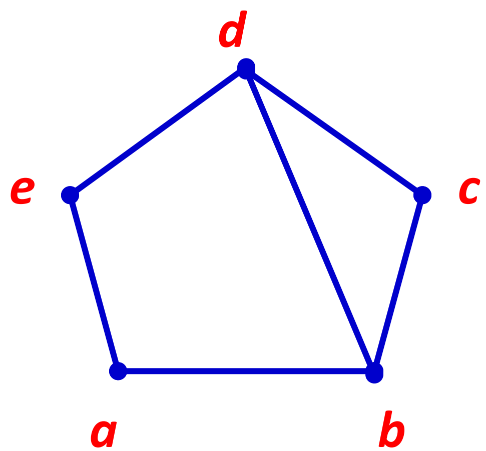
Graph *H*



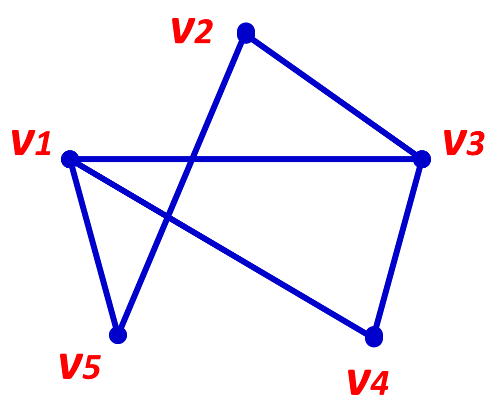
- Graph *G* and *H*, both have 6 vertices and 8 edges.
- Each has 4 vertices of degree three, and two vertices of degree two.
- *H* has a simple circuit of size three but all simple circuits in *G* have length at least four. So, *G* and *H* are not isomorphic.

Check Point

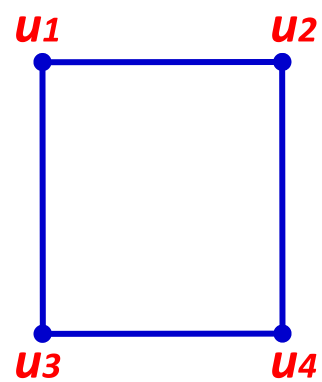
Determine if the following graph are isomorphic.



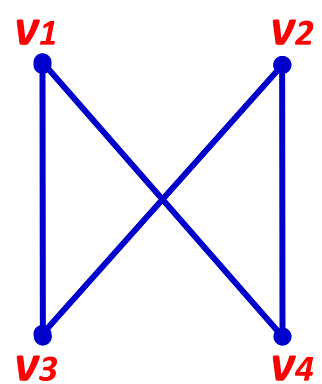
G1



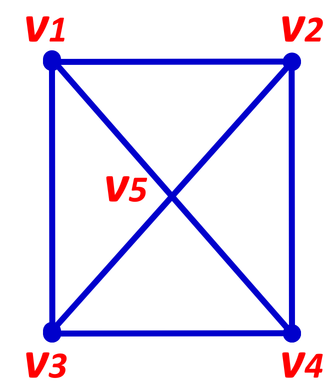
G2



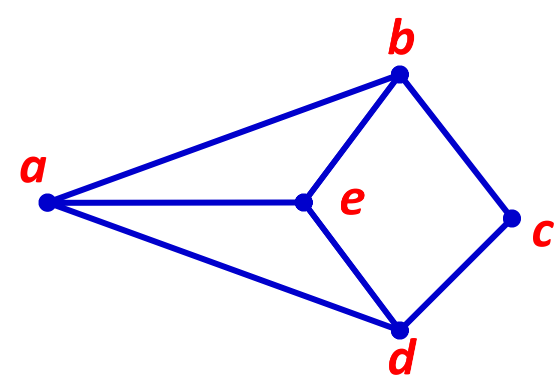
G



H



G1



G2

Counting Paths between Vertices

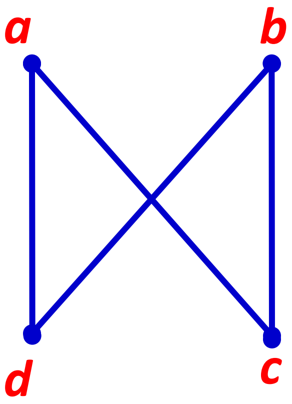
- Let G be a graph with adjacency matrix **A** with respect to the ordering $v1$, $v2$, …, vn (with directed or undirected edges, with multiple edges or loops allowed).

  The number of different paths of length r from vi to vj, where r is a positive integer, equals the (i, j)th entry of $\mathbf{A}^r$.

How many paths of length 4 are there from a to d in G?

Example

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \qquad \mathbf{A}^4 = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix}$$



Graph G

Hence, there are 8 paths of length 4 from a to d.

Check Point

Find the number of paths of length 2 between a to c of the following graphs.

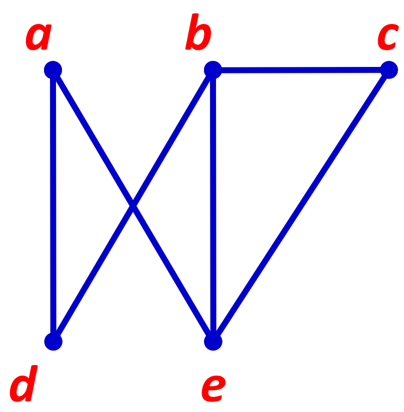


Figure 1

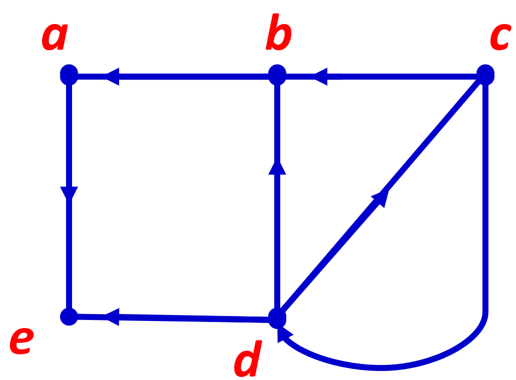
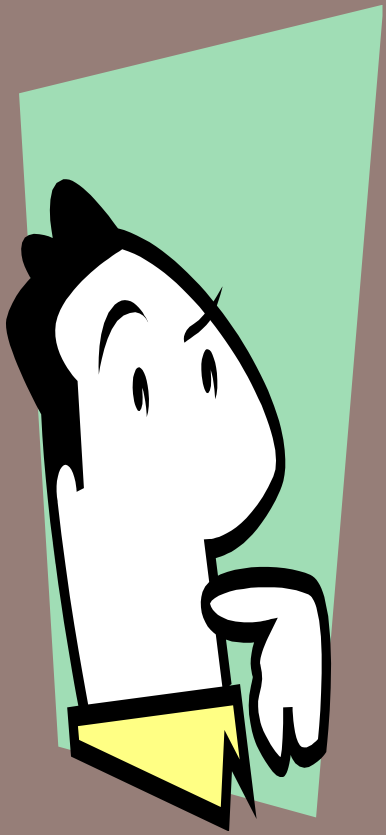


Figure 2

4. Euler And Hamiltonian Paths



- Determine Euler paths and Euler circuits in a graph
- Determine Hamiltonian paths and Hamiltonian circuits in a graph

4.A Euler's Paths & Circuits

Introduction

1. Can we travel along the edges of a graph starting at a vertex and returning to it by traversing each edge of the graph exactly one?

   - Solve by Euler circuit by examining the degree of vertices

2. Can we travel along the edges of a graph starting at a vertex and returning to it while visiting each vertex of the graph exactly one?

   - Solve by Hamiltonian circuit by examining the degree of vertices

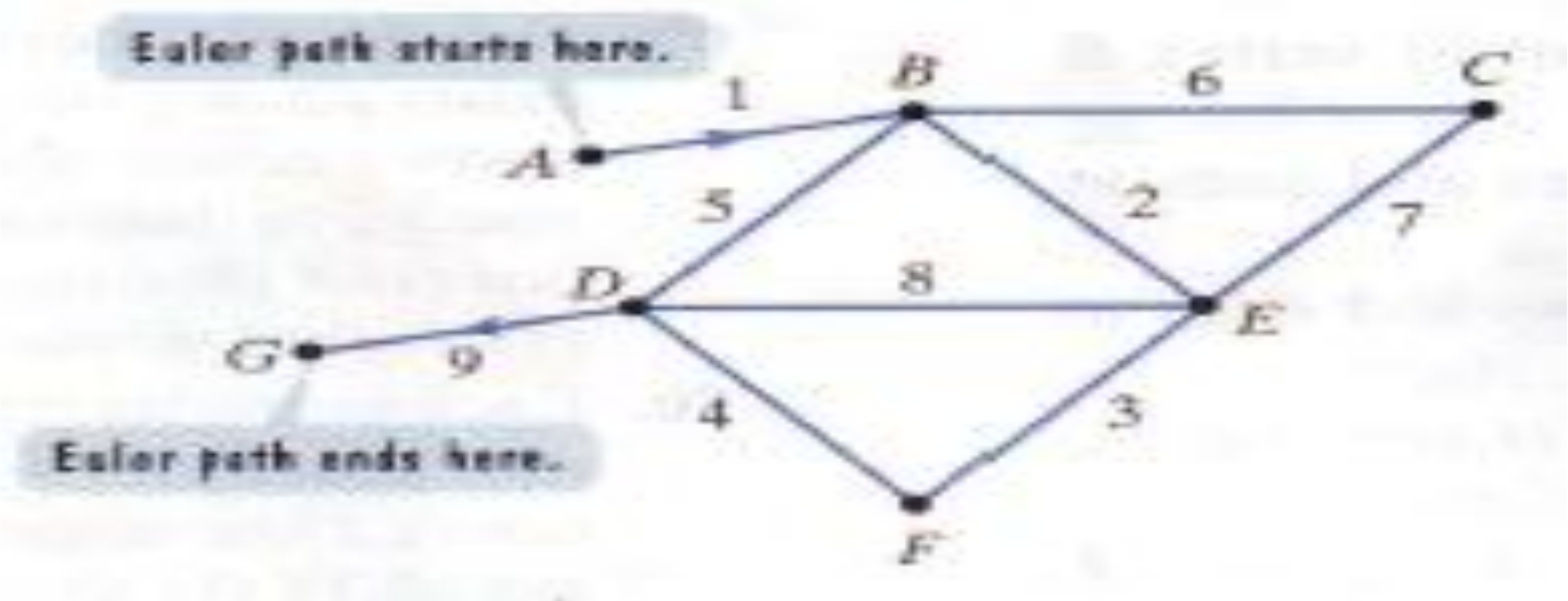
Euler Paths and Circuits

- Discovered by Swiss Mathematician Leonhard Euler (1736).

- An Euler path in a graph *G* is a simple path containing every edge of *G* (Travels through *every edge* once and only once - Each edge must be traveled and no edge can be retraced.)

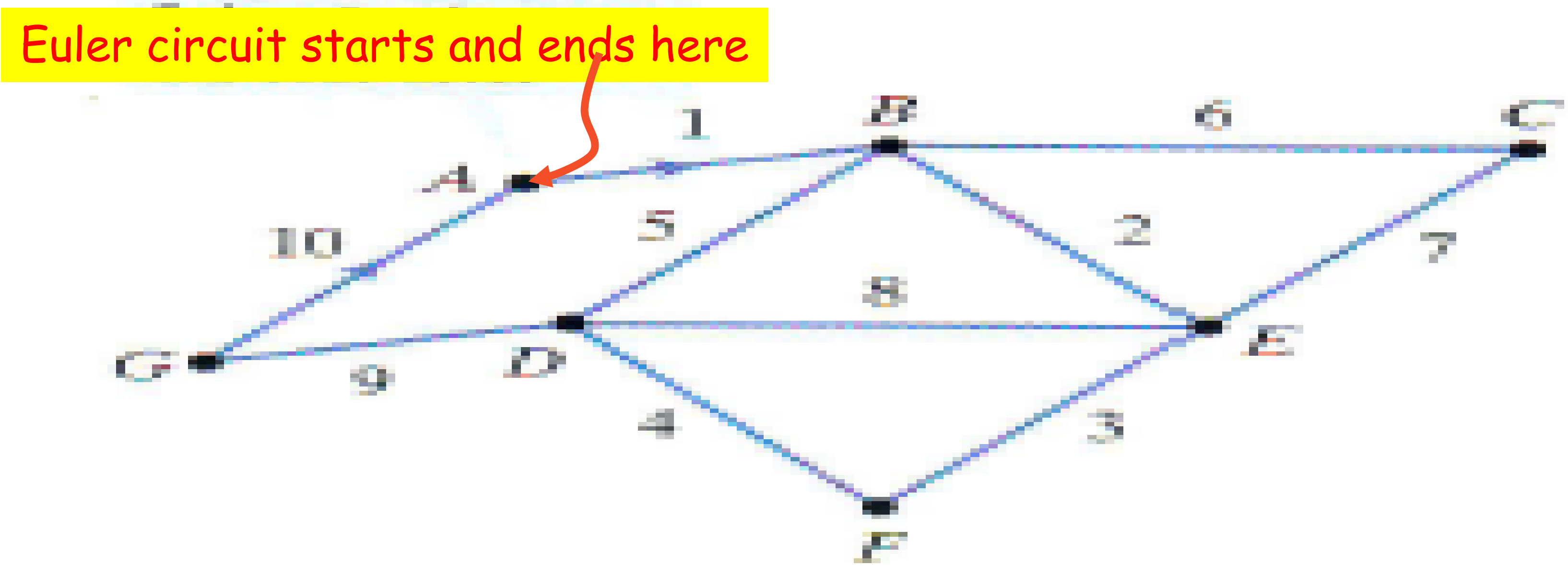
- An Euler circuit in a graph *G*: is a simple circuit containing every edge of *G* (Travels through *every edge* once and only once - Like all circuits, must begin and end at the same vertex.)

Euler Paths and Circuits



The path *A, B, E, F, D, B, C, E, D, G* is an Euler path because each edge is traveled once. Trace this path with your pencil. Now try using the numbers along the edges. The voice balloon indicates a starting vertex, *A,* and the arrow shows which way to trace first. When you arrive at the next vertex, *B,* take the next numbered edge, 2. When you arrive at the next vertex, *E,* take the next numbered edge, 3. Continue in this manner until numbered edge 9 ends the path at vertex *G*.

Euler Paths and Circuits



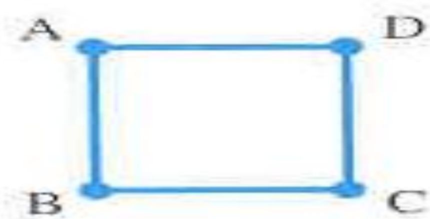
The path *A, B, E, F, D, B, C, E, D, G, A,* shown with numbered edges 1 through 10, is an Euler circuit. Do you see why? Each edge is traced only once. Furthermore, the path begins and ends at the same vertex, *A.* Notice that **every Euler circuit is an Euler path.** However, **not every Euler path is an Euler circuit.**

Some graphs have no Euler paths. Other graphs have several Euler paths. Furthermore, some graphs with Euler paths have no Euler circuits. **Euler's Theorem** is used to determine if a graph contains Euler paths or Euler circuits.

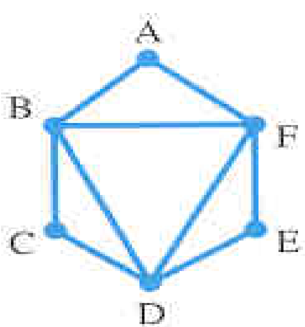
Check Point

a graph is shown and some sequences of vertices are specified. Determine which of these sequences show Euler circuits. If not, explain why not.

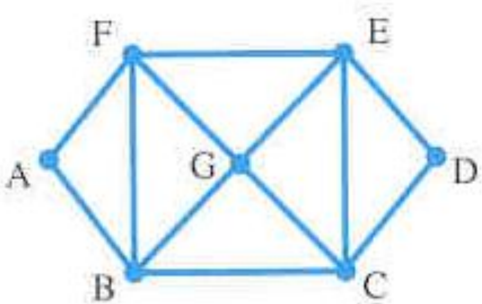
1. (a) $A \to B \to C \to D \to A \to B \to C \to D \to A$
   (b) $C \to B \to A \to D \to C$
   (c) $A \to C \to D \to B \to A$
   (d) $A \to B \to C \to D$



2. (a) $A \to B \to C \to D \to E \to F \to A$
   (b) $F \to B \to D \to B$
   (c) $A \to B \to C \to D \to E \to F \to B \to D \to F \to A$
   (d) $A \to B \to F \to D \to B \to C \to D \to E \to F \to A$



3. (a) $A \to B \to C \to D \to E \to F \to A$
   (b) $A \to B \to C \to D \to E \to G \to C \to E \to F \to G \to B \to F \to A$
   (c) $A \to B \to C \to D \to E \to C \to G \to E \to F \to G \to E \to F \to A$
   (d) $A \to B \to G \to E \to D \to C \to G \to F \to B \to C \to E \to F \to A$



Euler Paths and Circuits

- Euler's Theorem :

1. If a graph has exactly two odd vertices, then it has at least one Euler path, but no Euler circuit. Each Euler path must start at one of the odd vertices and end at the other one.
2. If a graph has no odd vertices (all even vertices), it has at least one Euler circuit (which, by definition, is also an Euler path). An Euler circuit can start and end at any vertex
3. If a graph has more than two odd vertices, then it has no Euler paths and no Euler circuits

- Necessary & Sufficient Conditions:
  - Theorem 1: A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.
  - Theorem 2: A connected multigraph has an Euler path but not an Euler circuit if and only if its has exactly two vertices of odd degree.

<u>Examples</u>: Euler Paths and Circuits

- Explain why the graph in Fig. I has at least one Euler path. Use trial and error to find one such path.

In the Fig. II, we count the number of edges at each vertex to determine if the vertex is odd or even. We see that there are exactly two odd vertices, namely *D* and *E*. By the first statement in Euler's Theorem, the graph has at least one Euler path, but no Euler circuit.

Euler's Theorem tells us that a possible Euler path must start at one of the odd vertices and end at the other one. We will use trial and error to determine an Euler path, starting at vertex *D* and ending at vertex *E*. Fig. III shows an Euler path: *D*, *C*, *B*, *E*, *C*, *A*, *B*, *D*, *E*. Trace this path and verify the numbers along the edges.

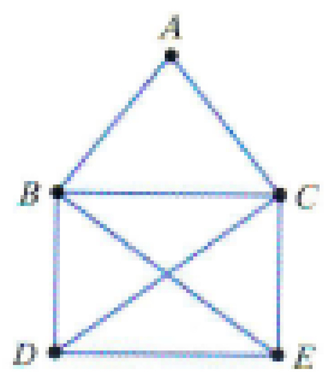


Fig. I

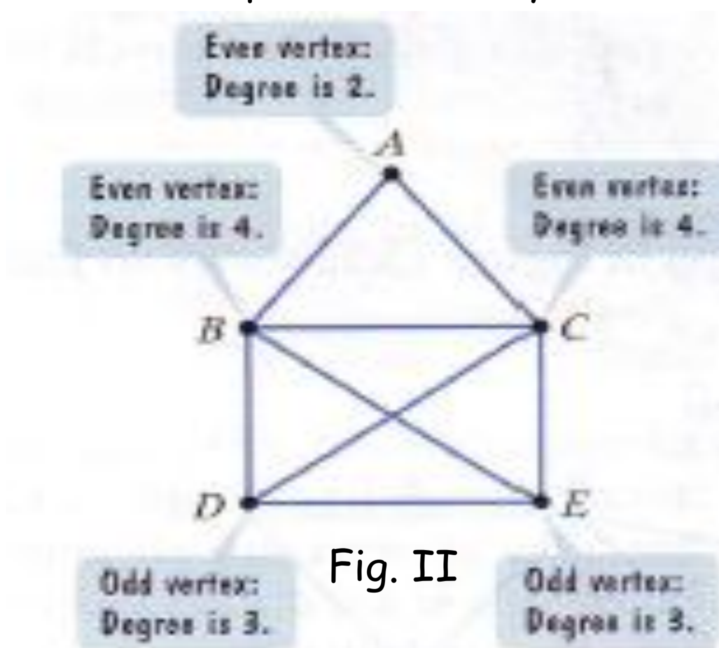


Fig. II

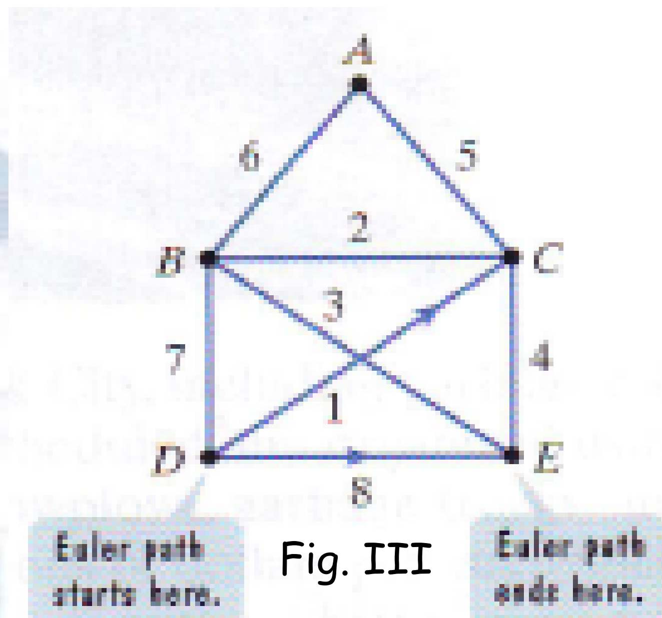


Fig. III

Examples: Euler Paths and Circuits

- Explain why the graph in Fig. A has at least one Euler circuit. **b.** Use trial and error to find one such circuit .

In Fig. B, we count the number of edges at each vertex to determine if the vertex is odd or even. We see that the graph has no odd vertices. By the second statement in Euler's Theorem, the graph has at least one Euler circuit

An Euler circuit can start and end at any vertex. We will use trial and error to determine an Euler circuit, starting and ending at vertex *H*. Remember that you must trace every edge exactly once and start and end at *H*. Fig. C shows an Euler circuit. Trace this circuit using the vertices in the figure's caption and verify the numbers along the edges

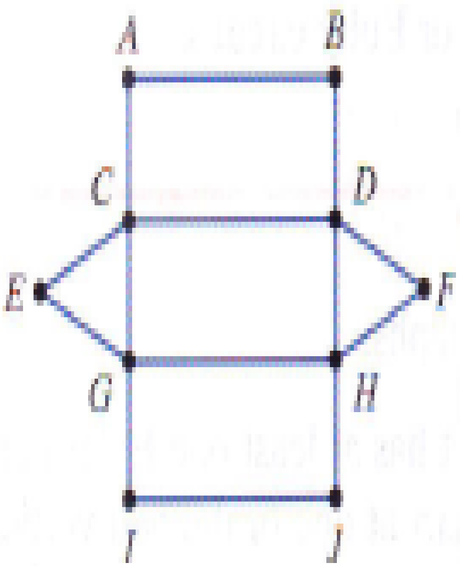


Fig. A

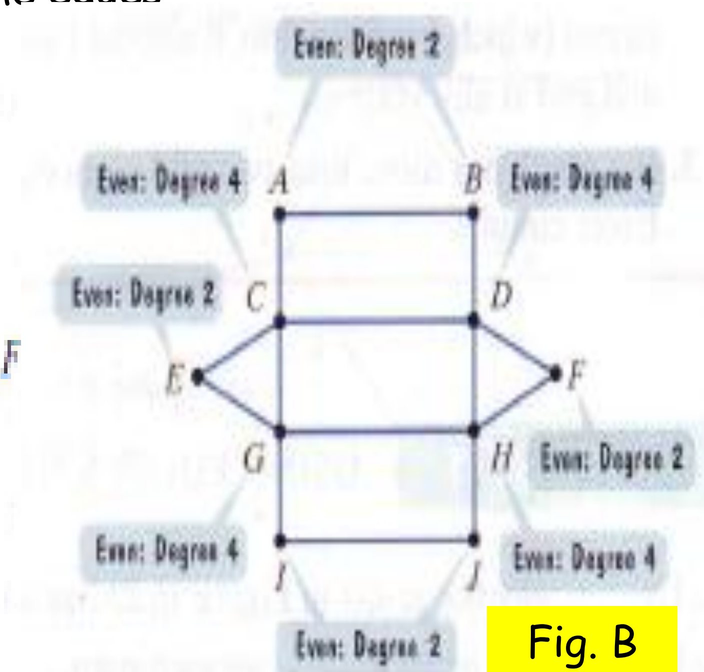


Fig. B

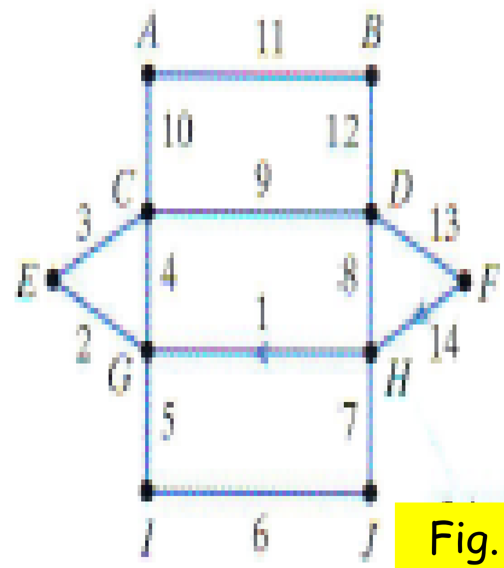
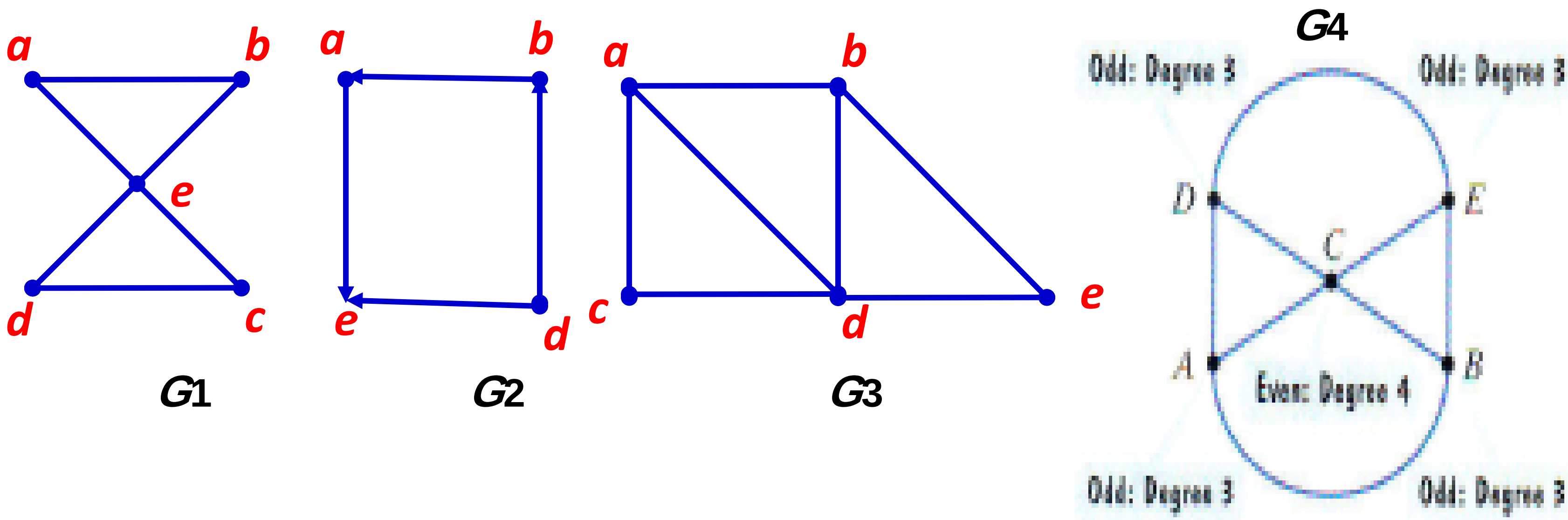


Fig. C

Examples: Euler Paths and Circuits



*G*1 : Euler Circuit: (*a*, *e*, *c*, *d*, *b*, *a*)

*G*2 : No Euler path and Euler circuit

*G*3: Euler Path: *G*3 (*a*, *c*, *d*, *e*, *b*, *d*, *a*, *b*)

*G*4: Shows a graph with one even vertex and four odd vertices. Because the graph has more than two odd vertices, by the third statement in Euler's Theorem, it has no Euler paths and no Euler circuits

Check Point

Use Euler's theorem to decide whether the graphs in figures 1-5 have an Euler circuit. (Do not actually find an Euler circuit.) Justify each answer briefly.

Fig. 1

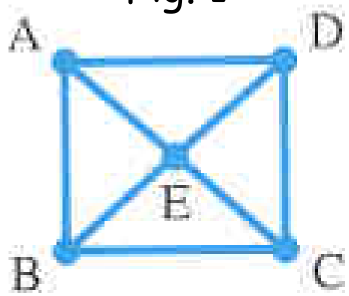


Fig. 2

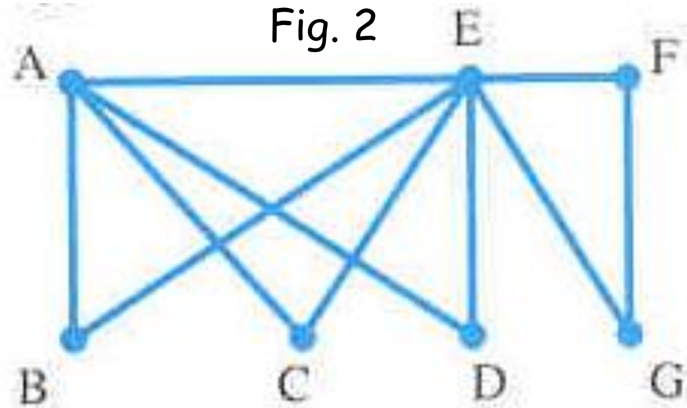
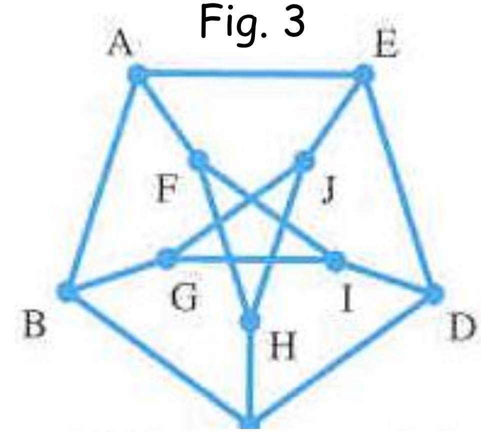


Fig. 3



c

Fig. 4

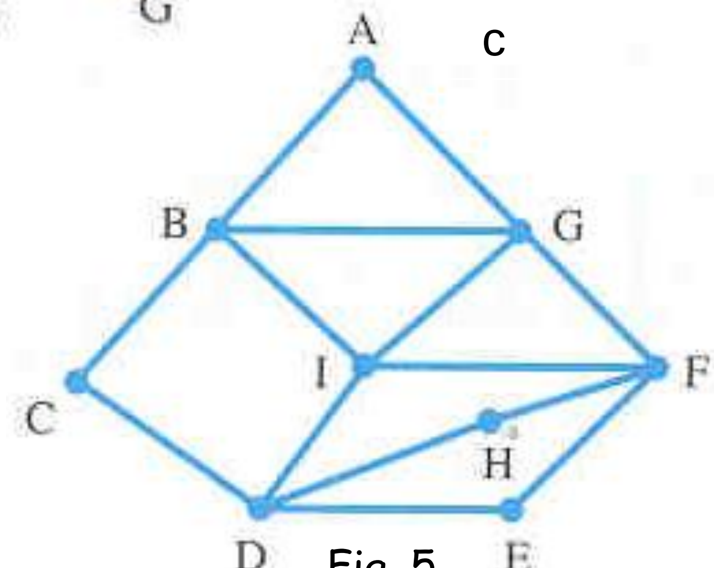


Fig. 5

Euler Paths and Circuits

Fluery's Algorithm

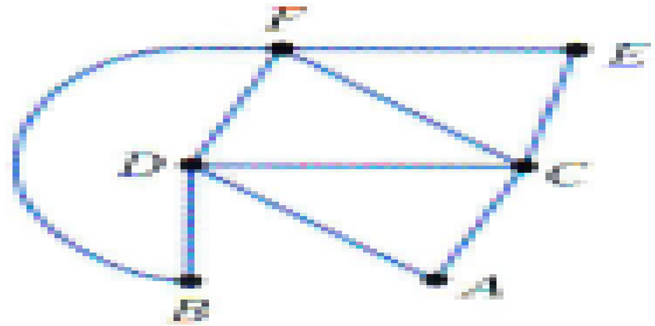
If Euler's theorem indicates the existence of an Euler path or Euler circuit, one can be found using the following procedure:

1. If the graph has exactly two odd vertices (and therefore an Euler path), choose one of the two odd vertices as the starting point. If the graph has no odd vertices (and therefore an Euler circuit), choose any vertex as the starting point.

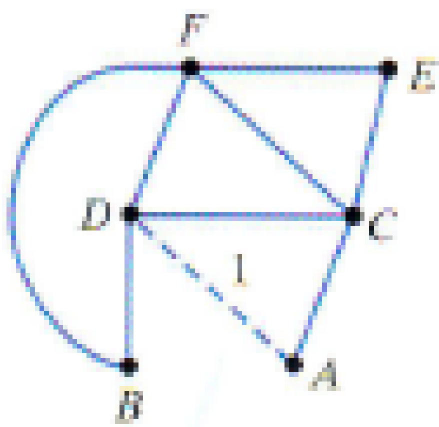
2. Number edges as you trace through the graph according to the following rules:
   - After you have traveled over an edge, erase it. (This is because you must travel each edge exactly once.) Show the erased edges as dashed lines.
   - When faced with a choice of edges to trace, choose an edge that is not a bridge. Travel over an edge that is a bridge only if there is no alternative.

Euler Paths and Circuits
Fluery's Algorithm

Use Fluery's Algorithm tob find an Euler's circuit for the graph shown in R.H.S.



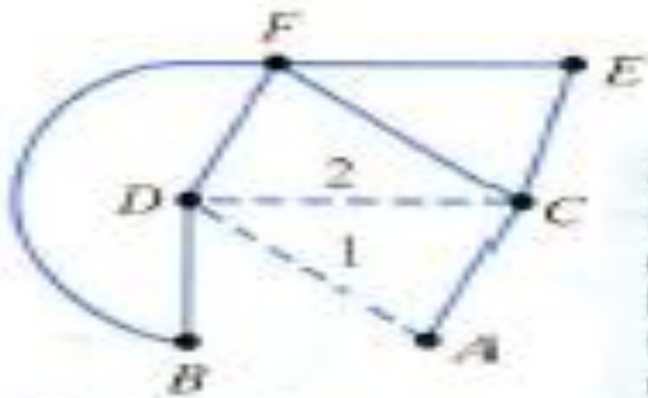
Step 1



Travel from A to D and erase edge AD.

We can also travel from A to C.

Step 2



CA is a bridge. If it were removed, vertex A would be isolated from the rest of the graph.

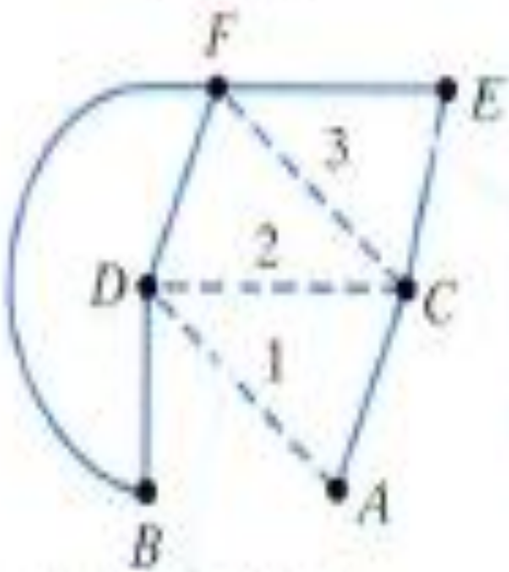
Travel from D to C and erase edge DC.

We can also travel from D to F or B.

Euler Paths and Circuits

Fluery's Algorithm

Step 3

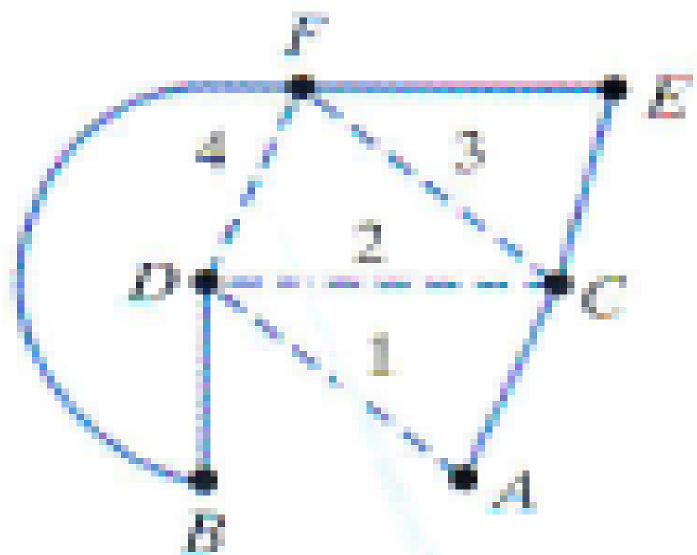


FE is a bridge. If it were removed, the graph would have two disconnected components.

Travel from *C* to *F* and erase edge *CF*.

We can also travel from *C* to *E*, but not from *C* to *A*. Don't cross the bridge.

Step 4

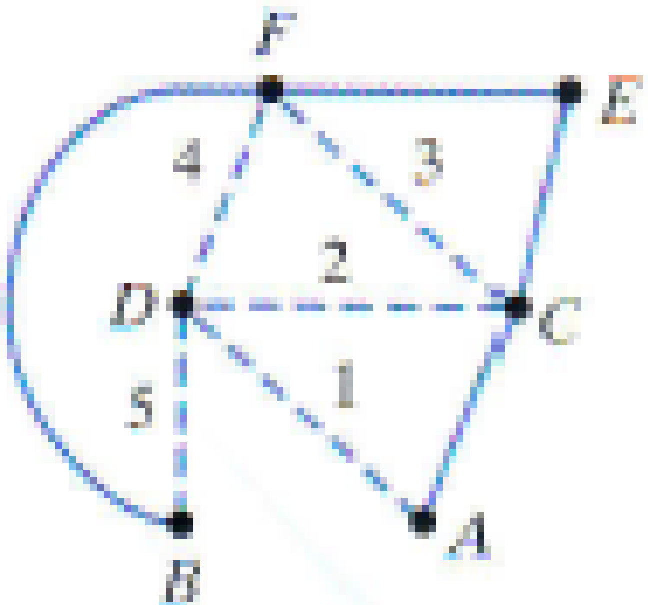


Travel from *F* to *D* and erase edge *FD*.

We can also travel from *F* to *B*, but not from *F* to *E*. Don't cross the bridge.

Euler Paths and Circuits
<u>Fluery's Algorithm</u>

Step 5



Travel from D to B and erase edge DB.

There are no other choices.

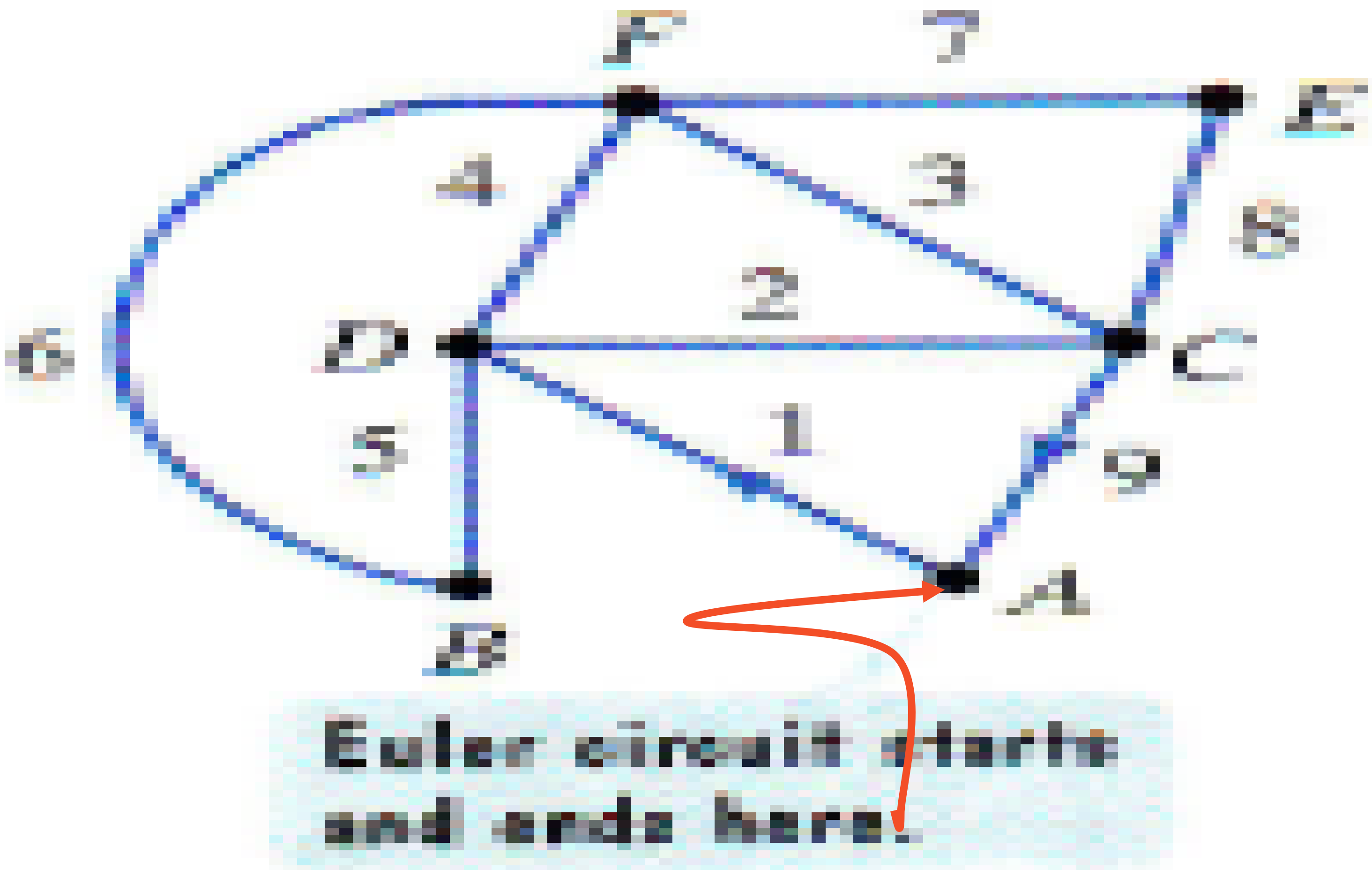
Steps 6, 7, 8, 9

Travel from B to F, F to E, E to C, and C to A, and erase the respective edges.

There are no choices at each step.

Euler Paths and Circuits
Fluery's Algorithm

The completed Euler circuit is shown in Figure below



Exercise Set 4.A

1-14. In Exercises 1-14 determine which of the following graphs have an Euler path, Euler Circuit and those that do not have? If the graph has an Euler path or circuit, use trial and error or Fleury's Algorithm to find one.

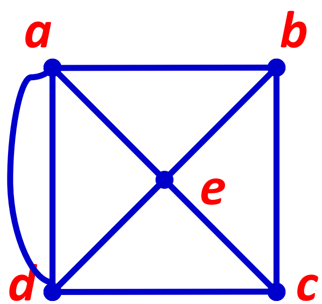


Figure 1

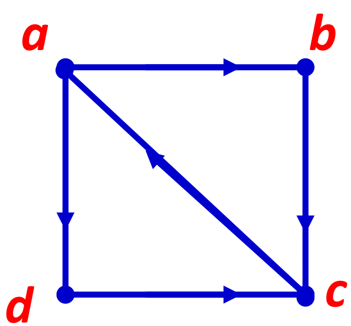


Figure 2

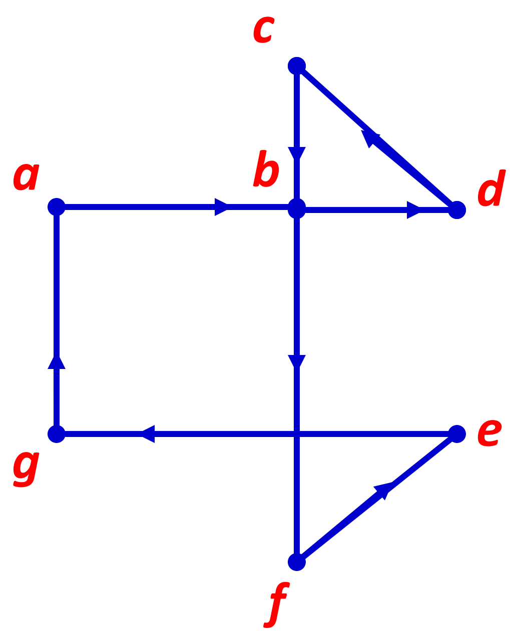


Figure 3

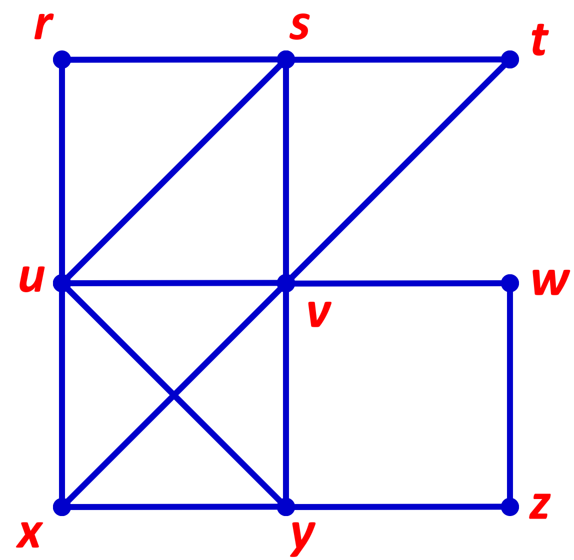


Figure 4

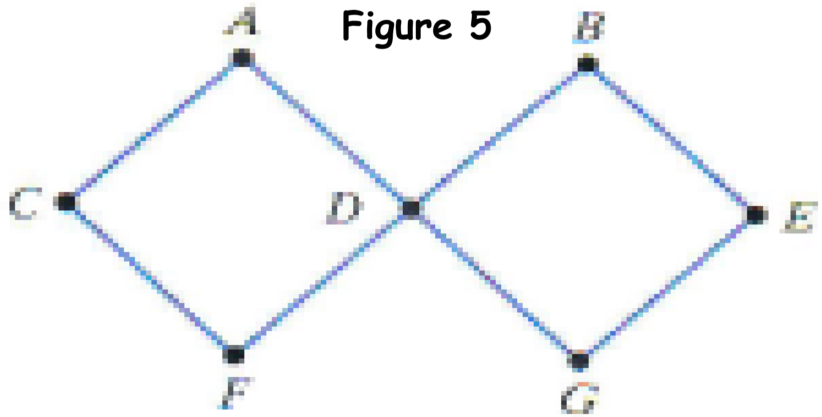


Figure 5

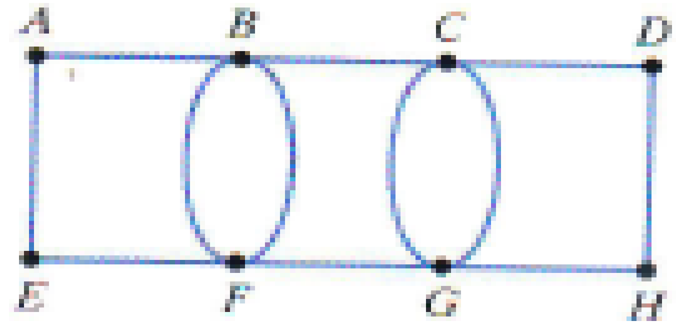
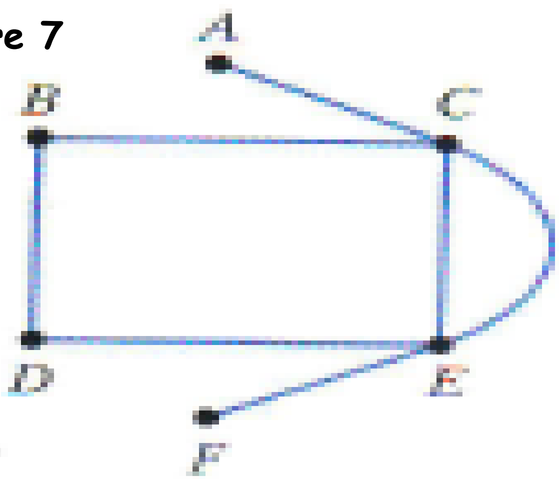


Figure 6

Figure 7



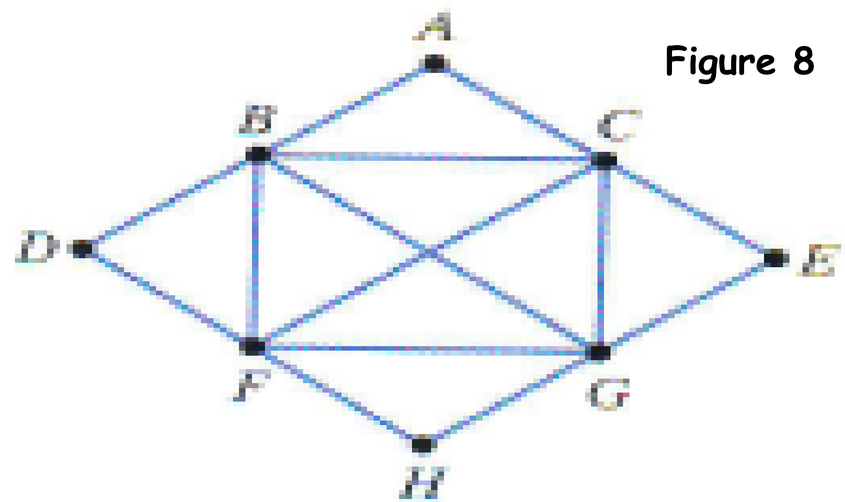


Figure 8

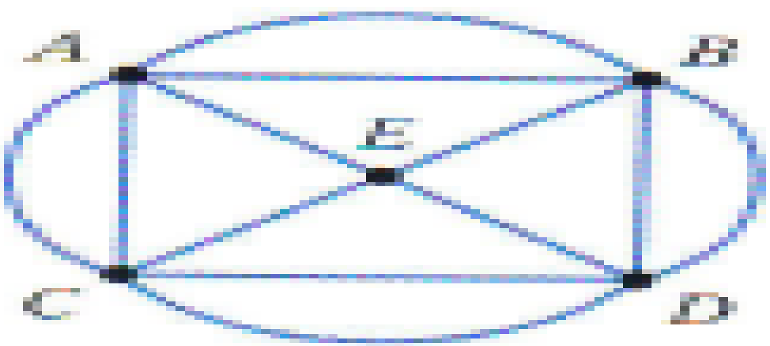


Figure 9

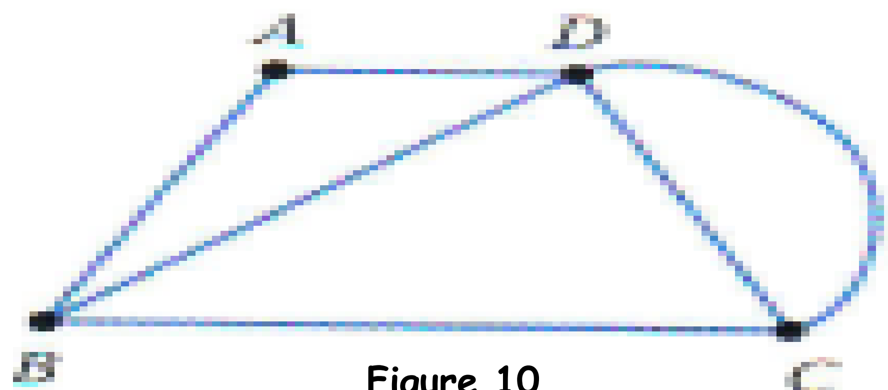
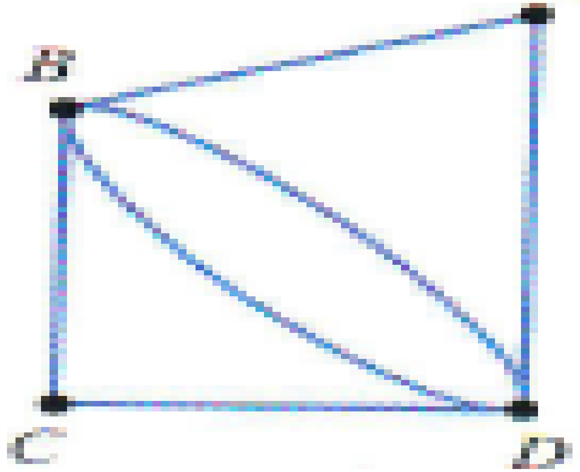


Figure 10

Figure 11



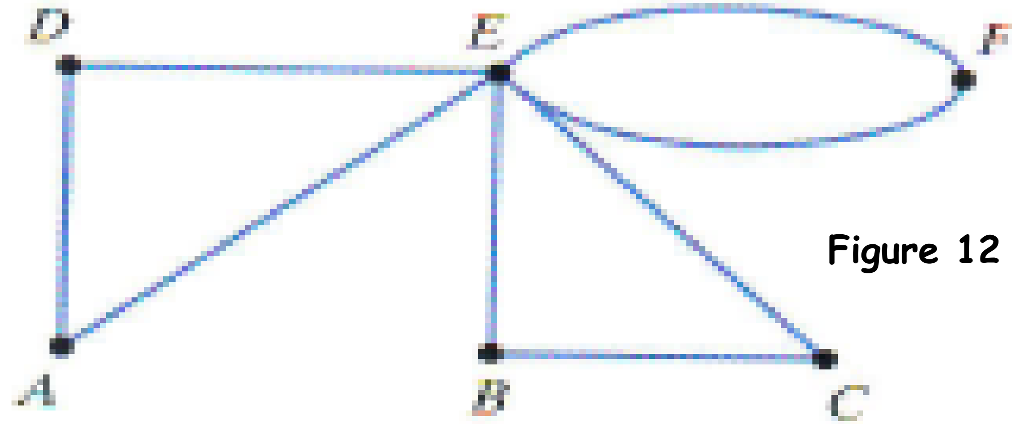


Figure 12

Figure 13

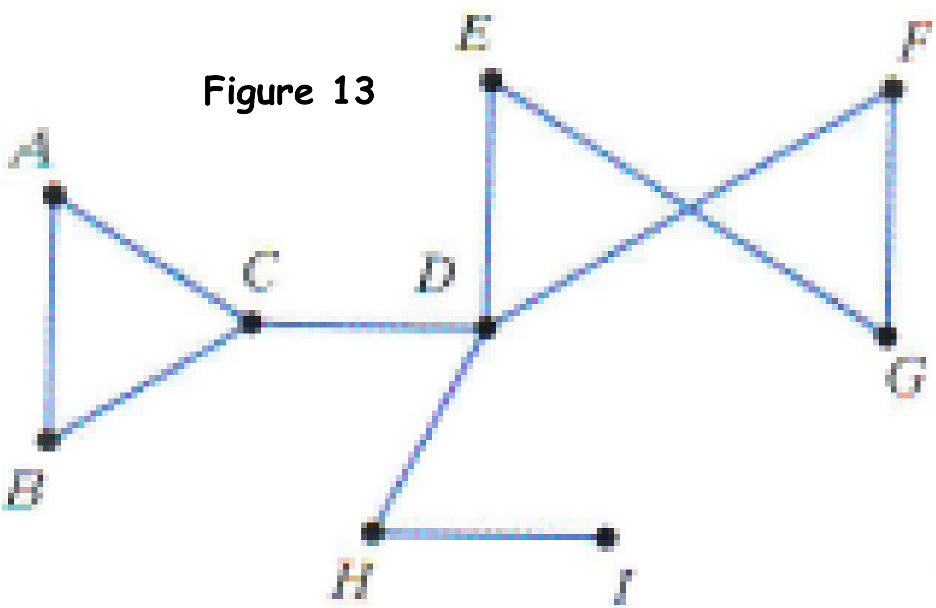
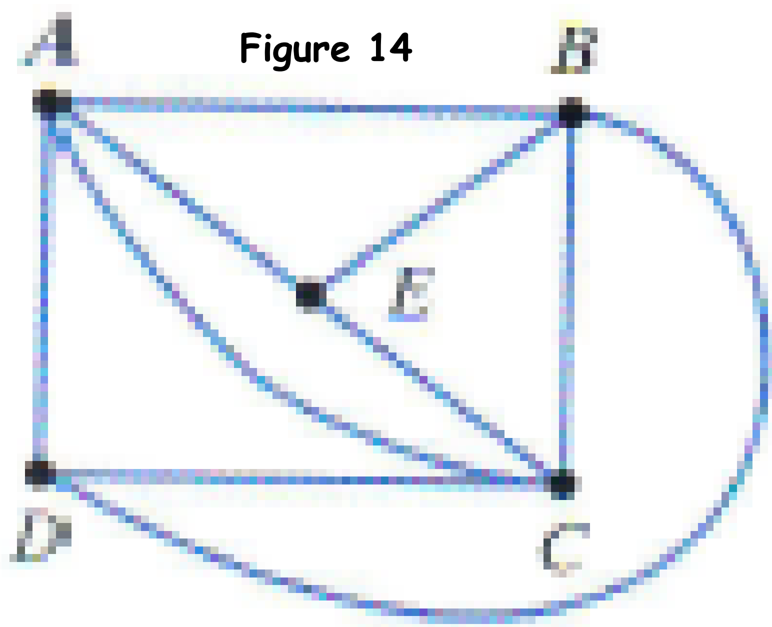


Figure 14



15-17. For Exercises 15-17, determine whether the graph has an Euler path that begins and ends at different vertices. Justify your answer. If the graph has such a path, say at which vertices the path must begin and end.

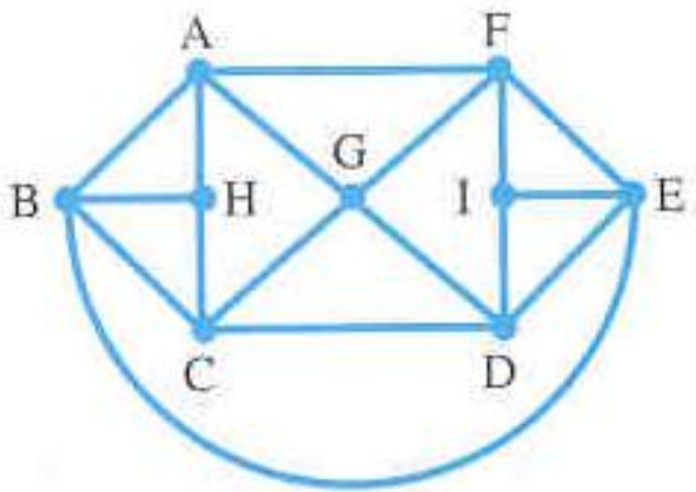


Figure 15

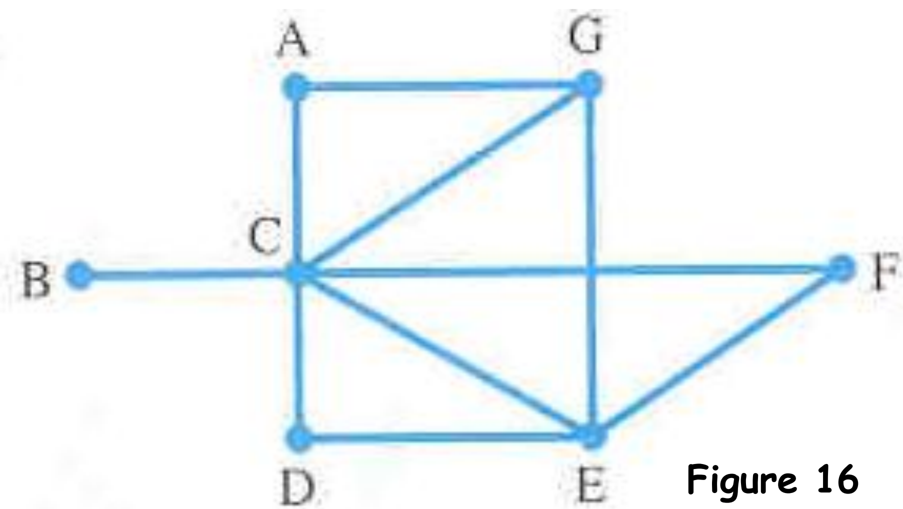
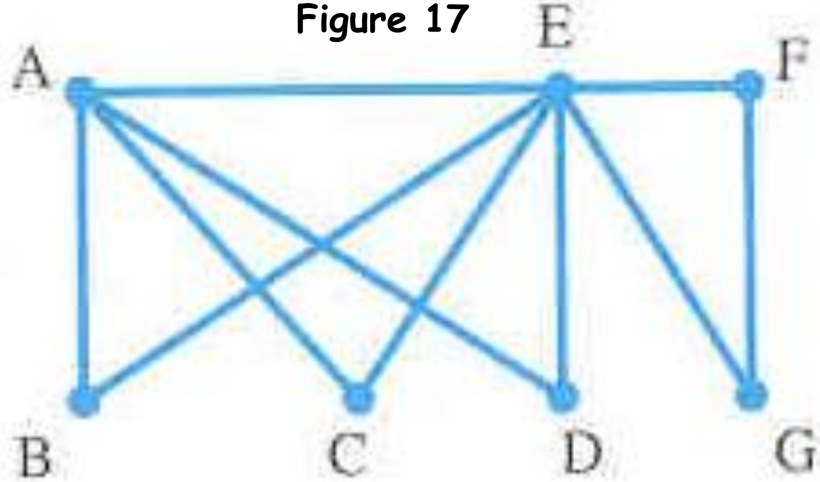


Figure 16

Figure 17



18-20. Use Euler's theorem to determine whether the given graph has an Euler circuit. If not, explain why not. If the graph does have an Euler circuit, use Fleury's algorithm to find an Euler circuit for the graph. There are many different correct answers

Figure 18

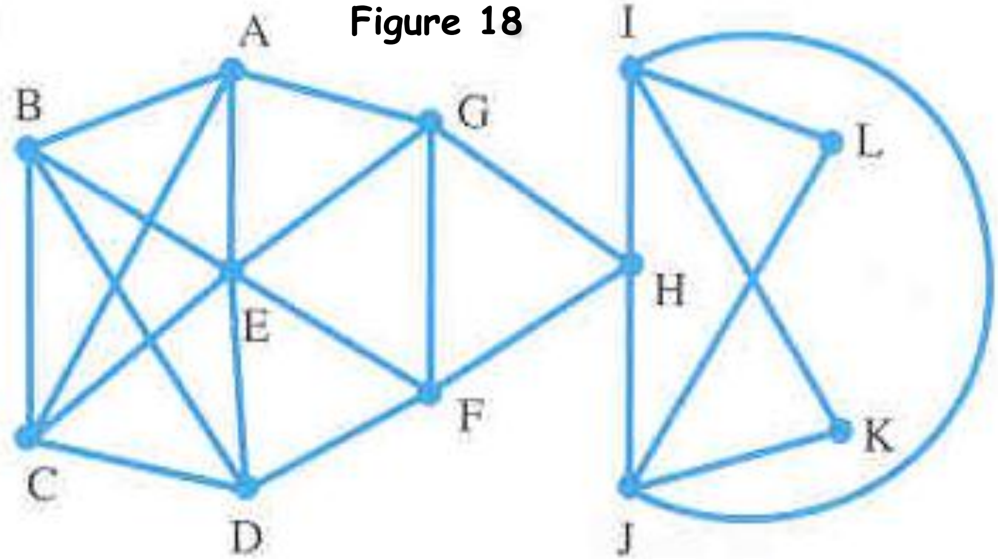
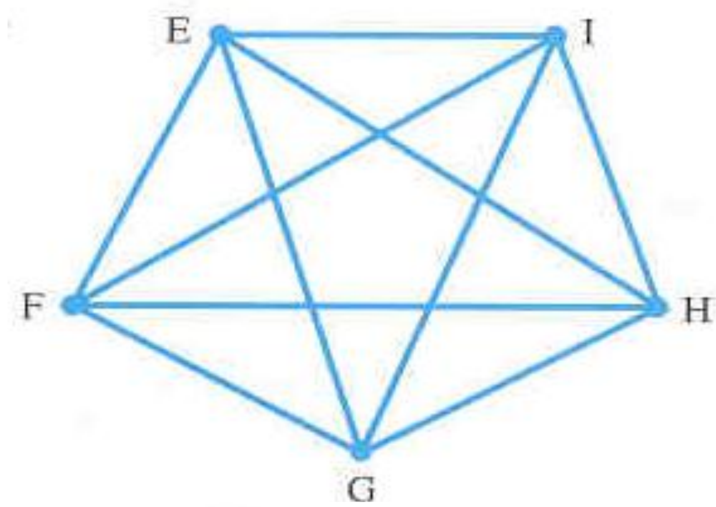


Figure 19



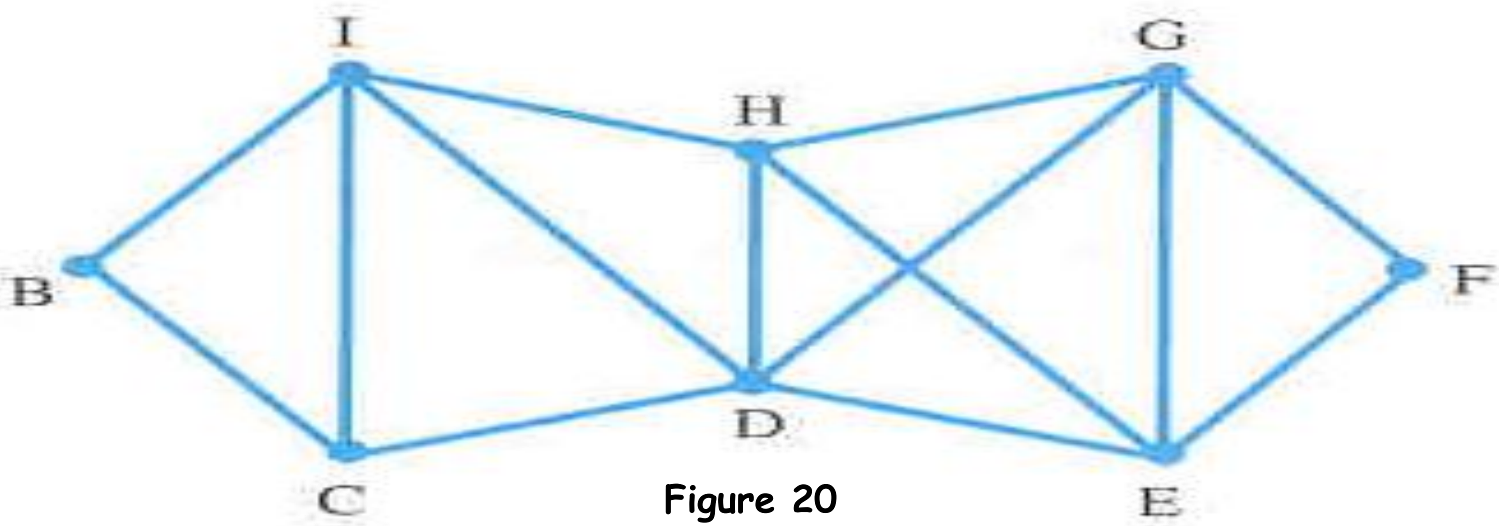


Figure 20

21-25. For Exercises 18-22 use Euler's theorem to determine whether the graph has an Euler circuit, justifying each answer. Then determine whether the graph has a circuit that visits each vertex exactly once, except that it returns to its starting vertex. If so, write down the circuit. (There is more than one correct answer for some of these.)

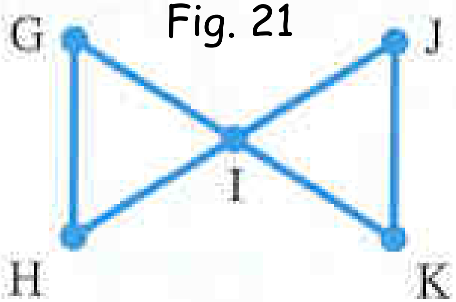


Fig. 21

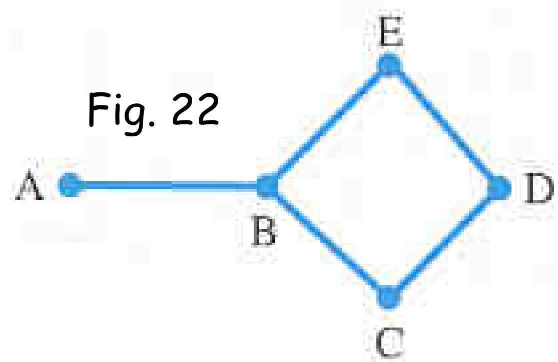


Fig. 22

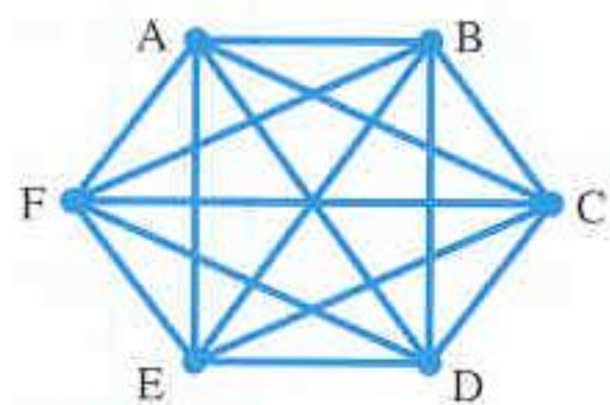


Fig. 23

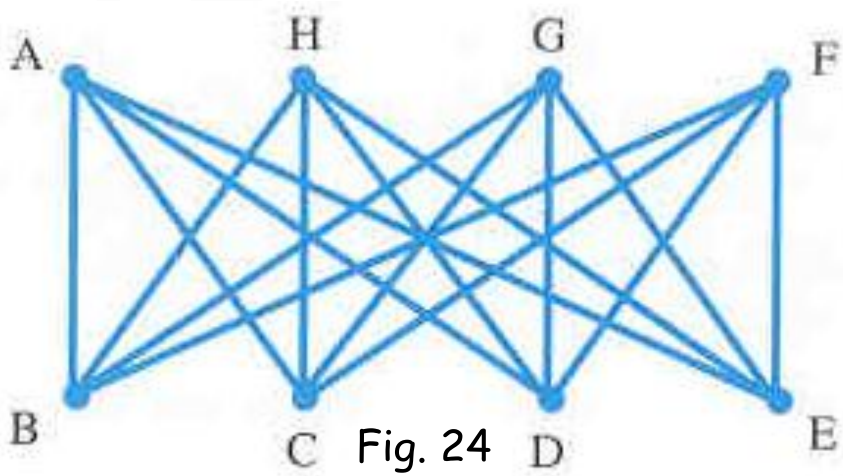


Fig. 24

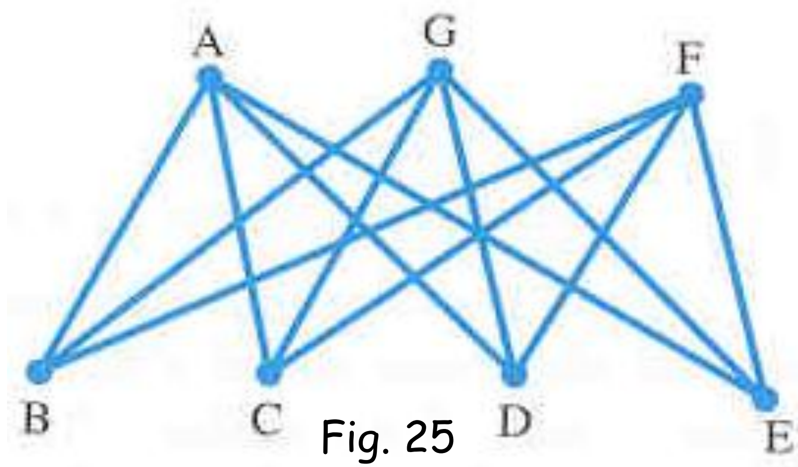


Fig. 25

26-29. In Exercises 23-26, use Fleury's Algorithm to find an Euler path.

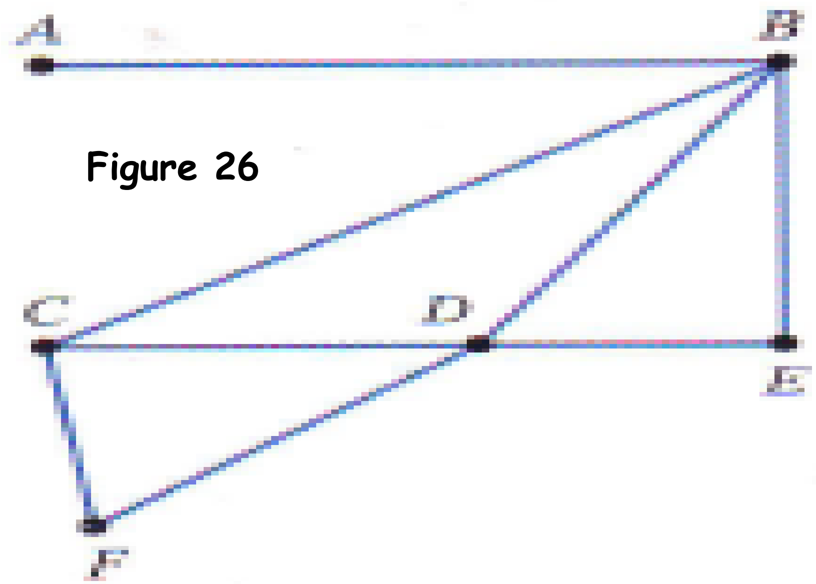


Figure 26

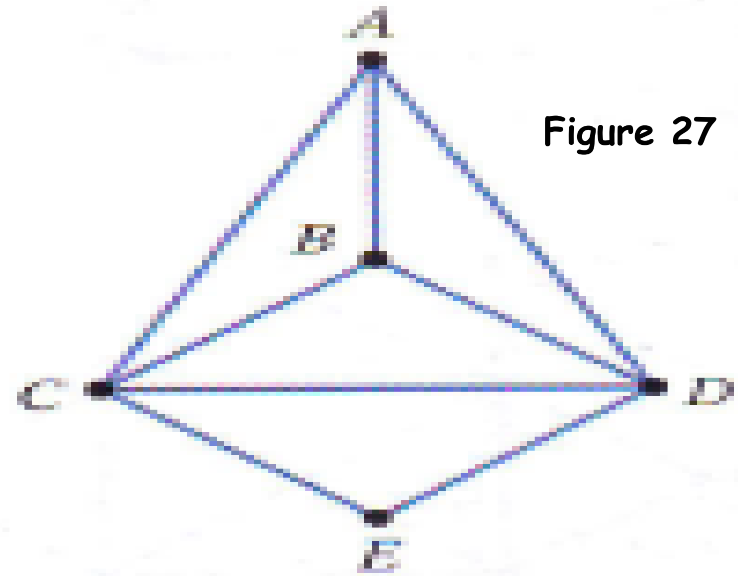


Figure 27

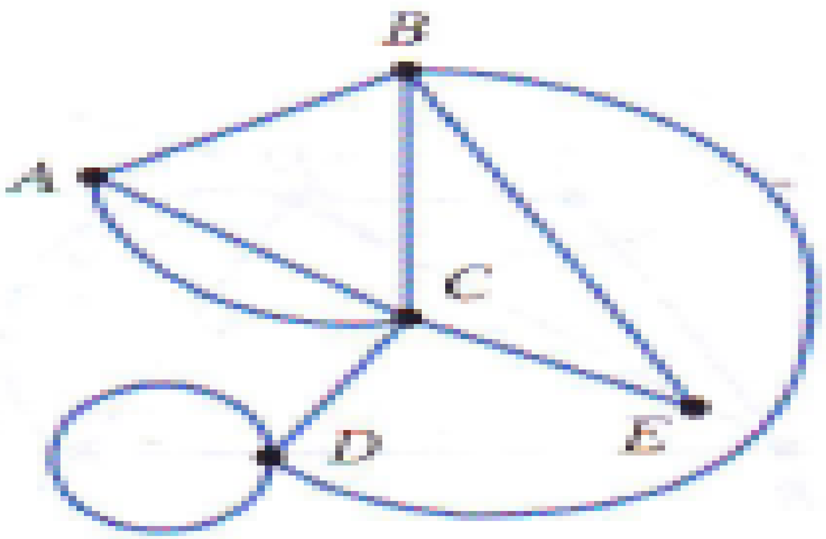


Figure 28

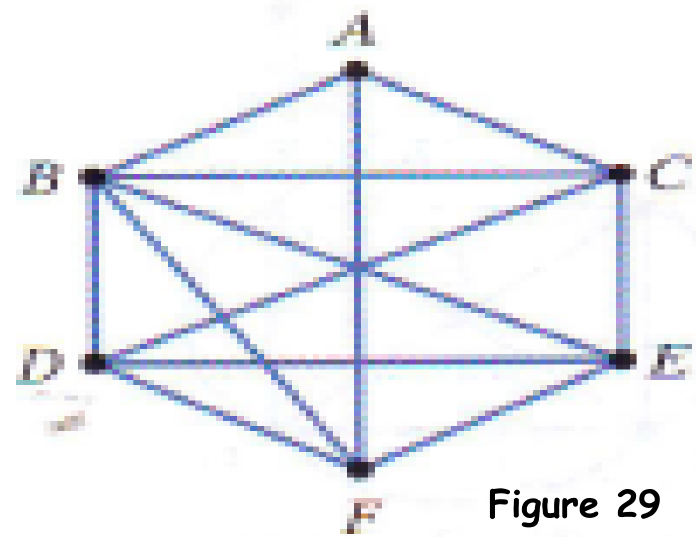


Figure 29

30-33. In Exercises 27-30, use Fleury's Algorithm to find an Euler circuit.

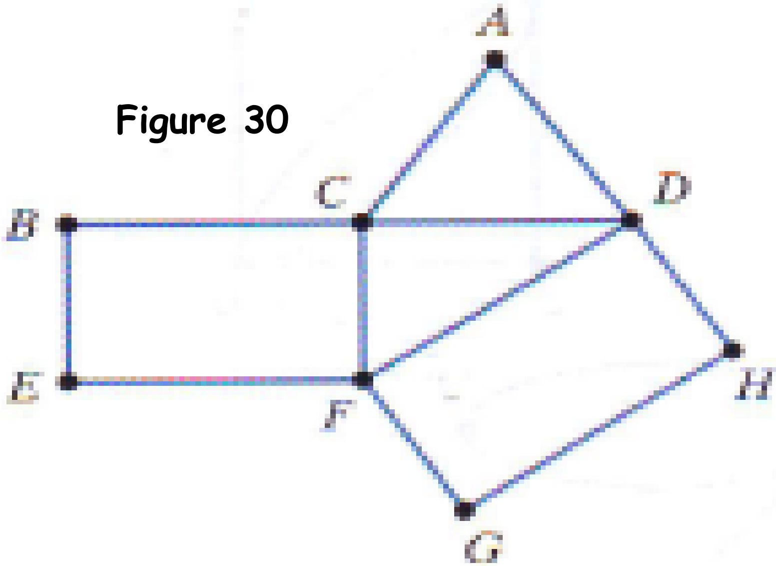


Figure 30

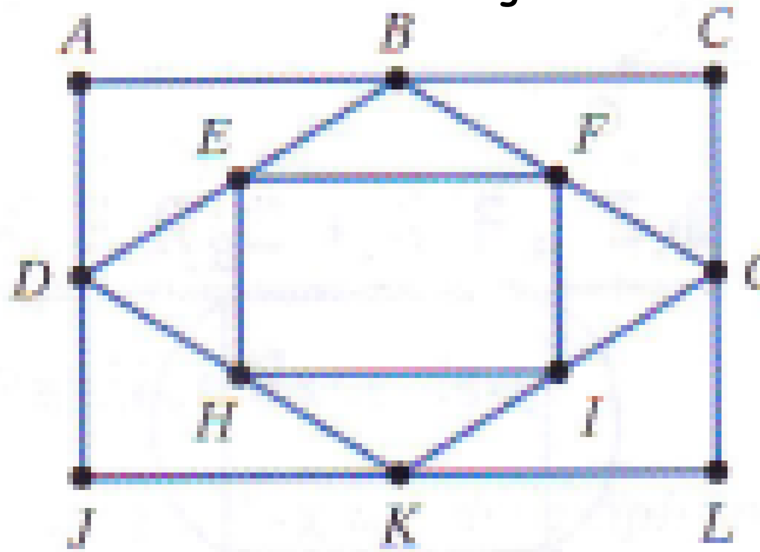


Figure 31

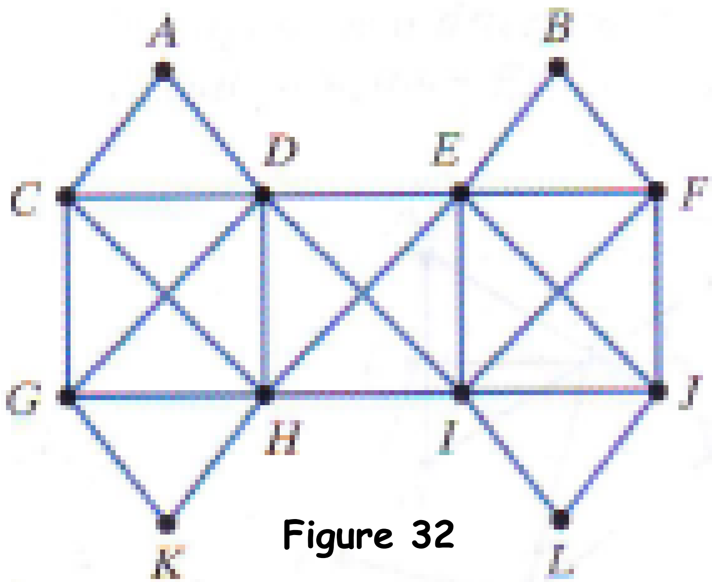


Figure 32

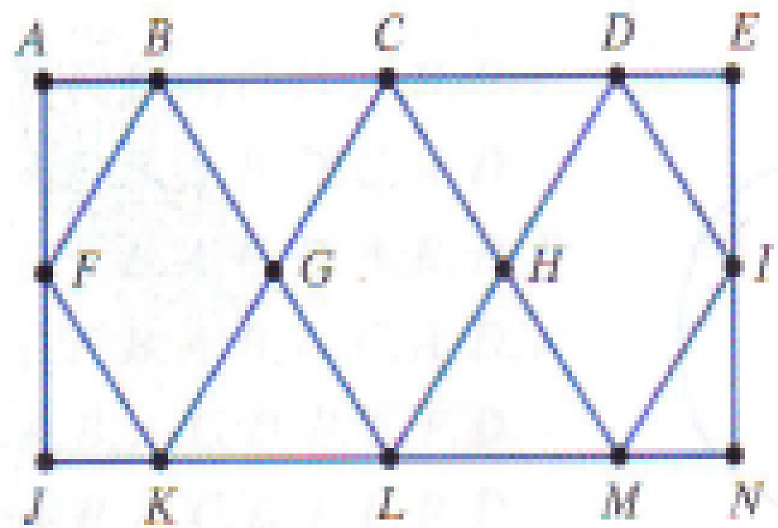


Figure 33

34-36. In Exercises 31-33, use Fleury's Algorithm to find an Euler circuit beginning and ending at A. There are many different correct answers.

Figure 34

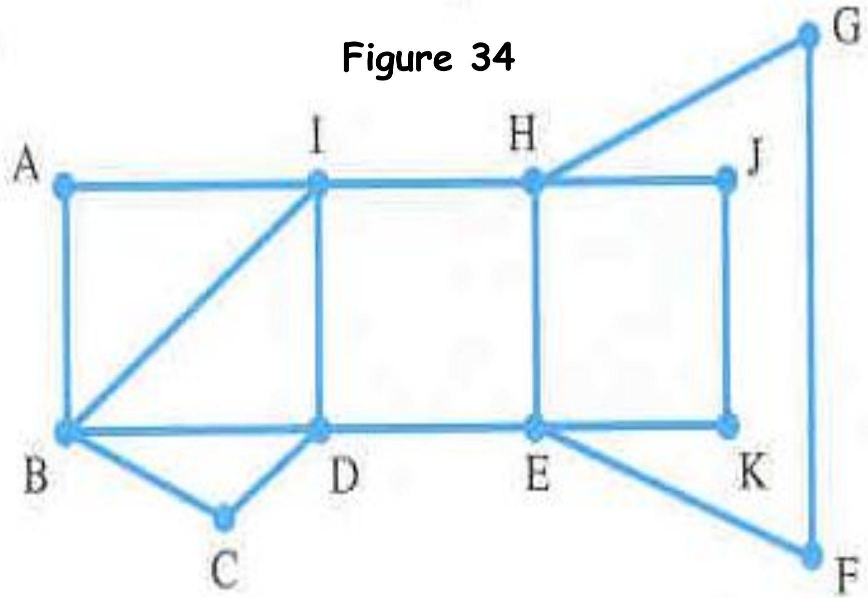
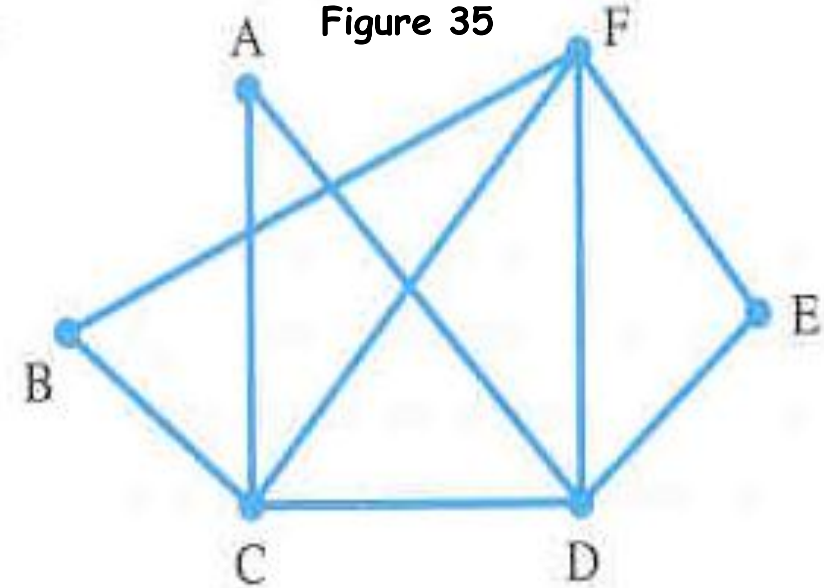


Figure 35



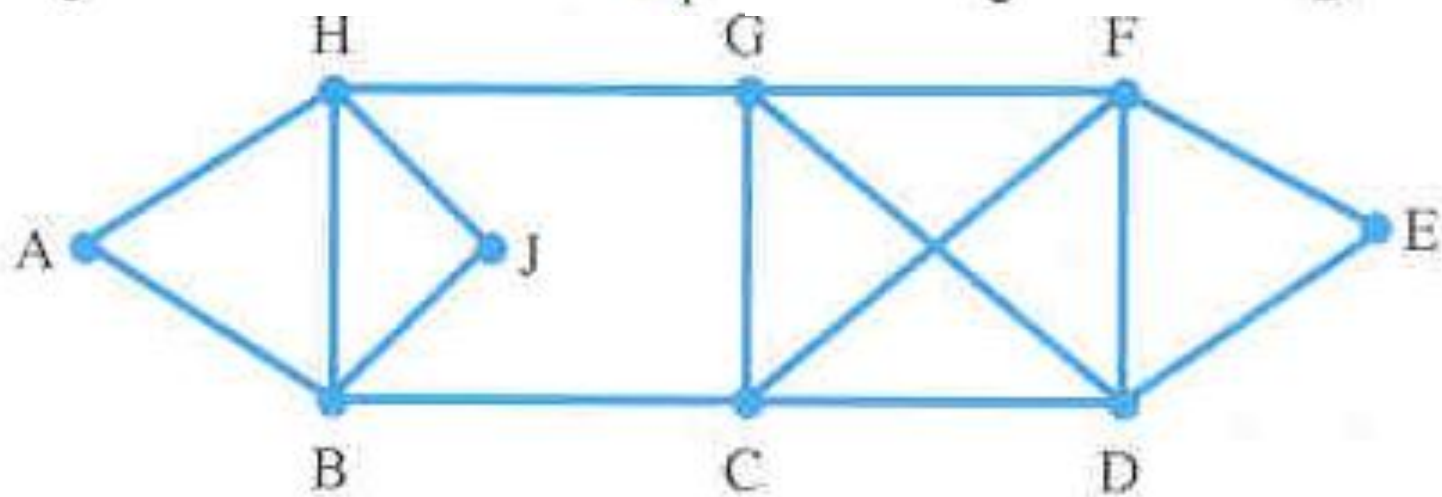


Figure 36

37-40 In Exercises 34-37, a graph is given:

a. Modify the graph by removing the least number of edges so that the resulting graph has an Euler circuit. b. Find an Euler circuit for the modified graph..

Figure 37

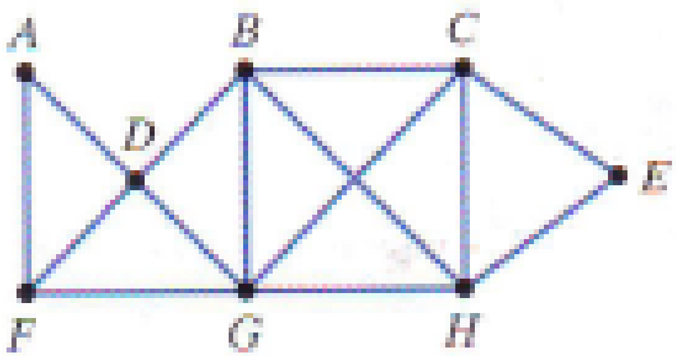
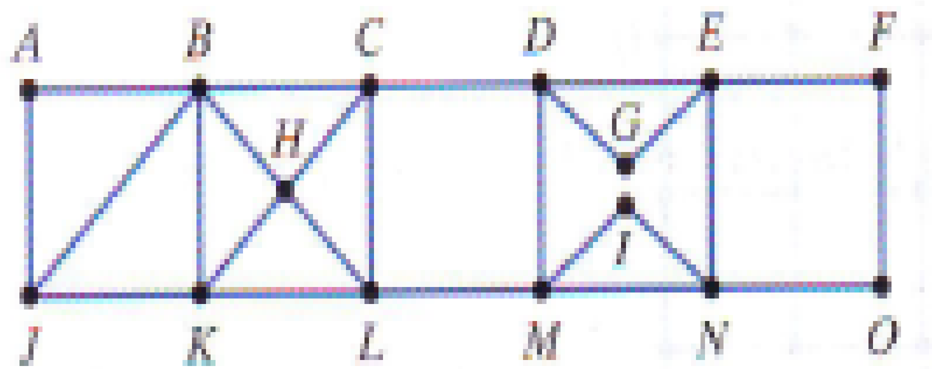


Figure 38



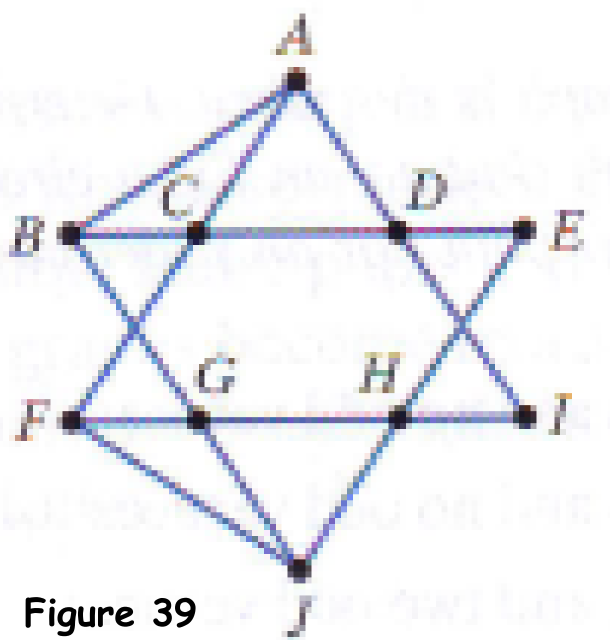


Figure 39

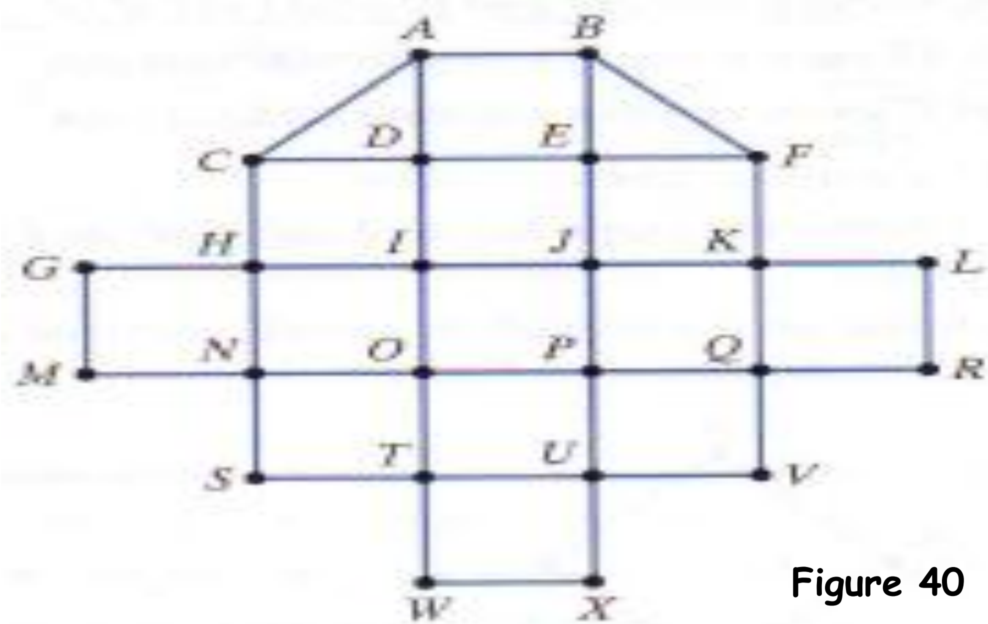


Figure 40

41-42 . For Exercises 41-42, use Euler's theorem to determine whether it is possible to begin and end at the same place, trace the pattern without lifting your pencil, and trace over no line in the pattern more than once.

Fig. 41

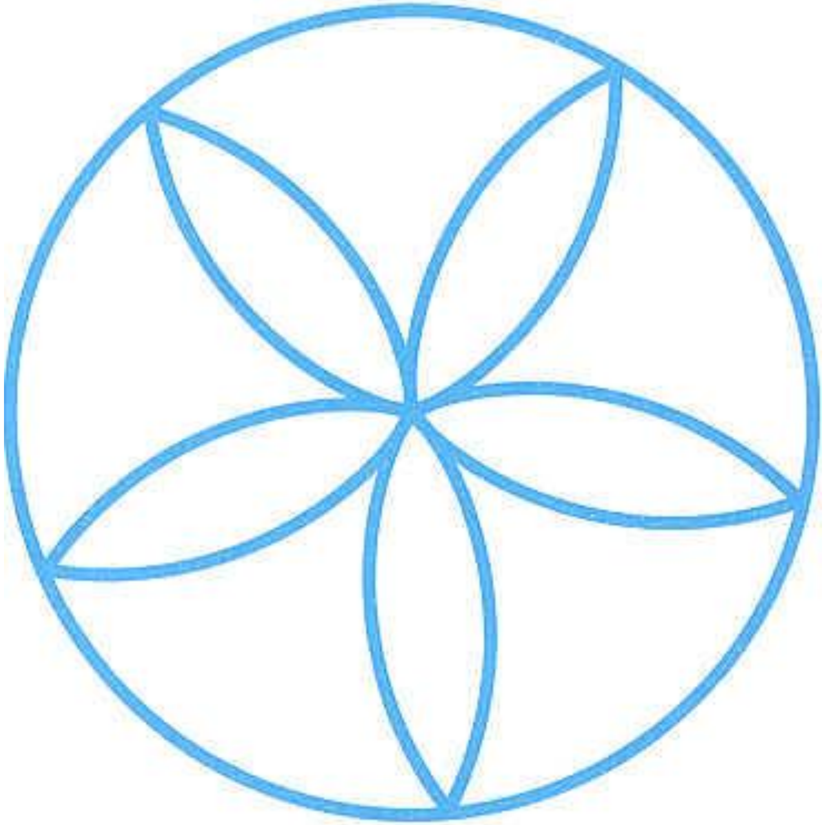
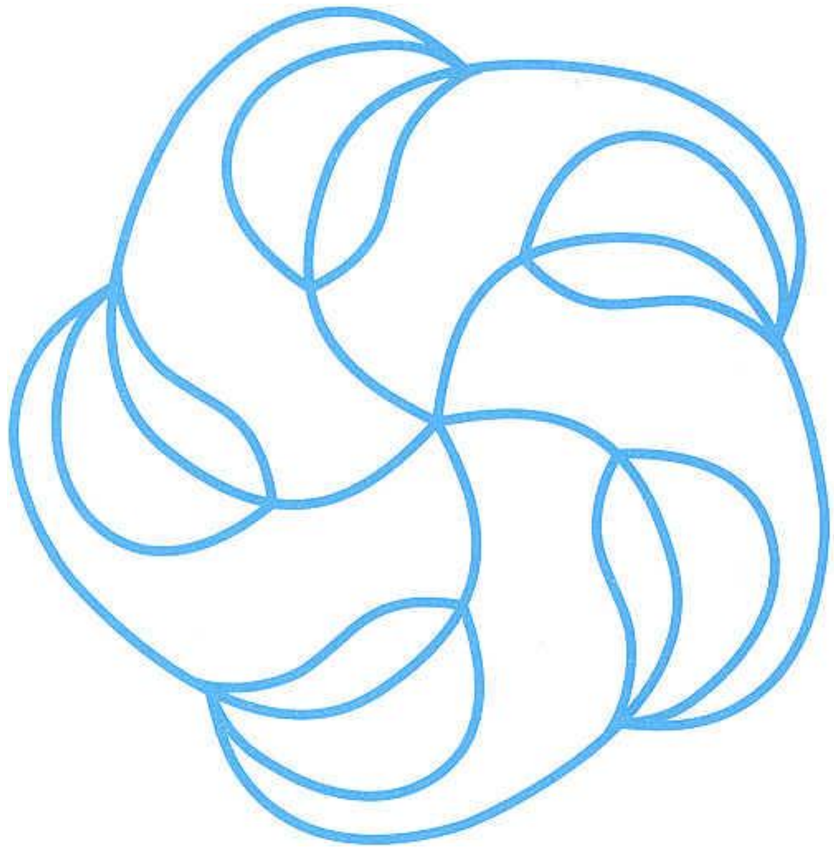


Fig. 42



43. The graph in the next column shows the layout of the paths in a botanical garden. The edges represent the paths. Has the garden been designed in such a way that it is possible for a visitor to find a route that. begins and ends at the entrance to the garden (represented by the vertex A) and that goes along each path exactly once? If so, use Fleury's algorithm to find such a route.

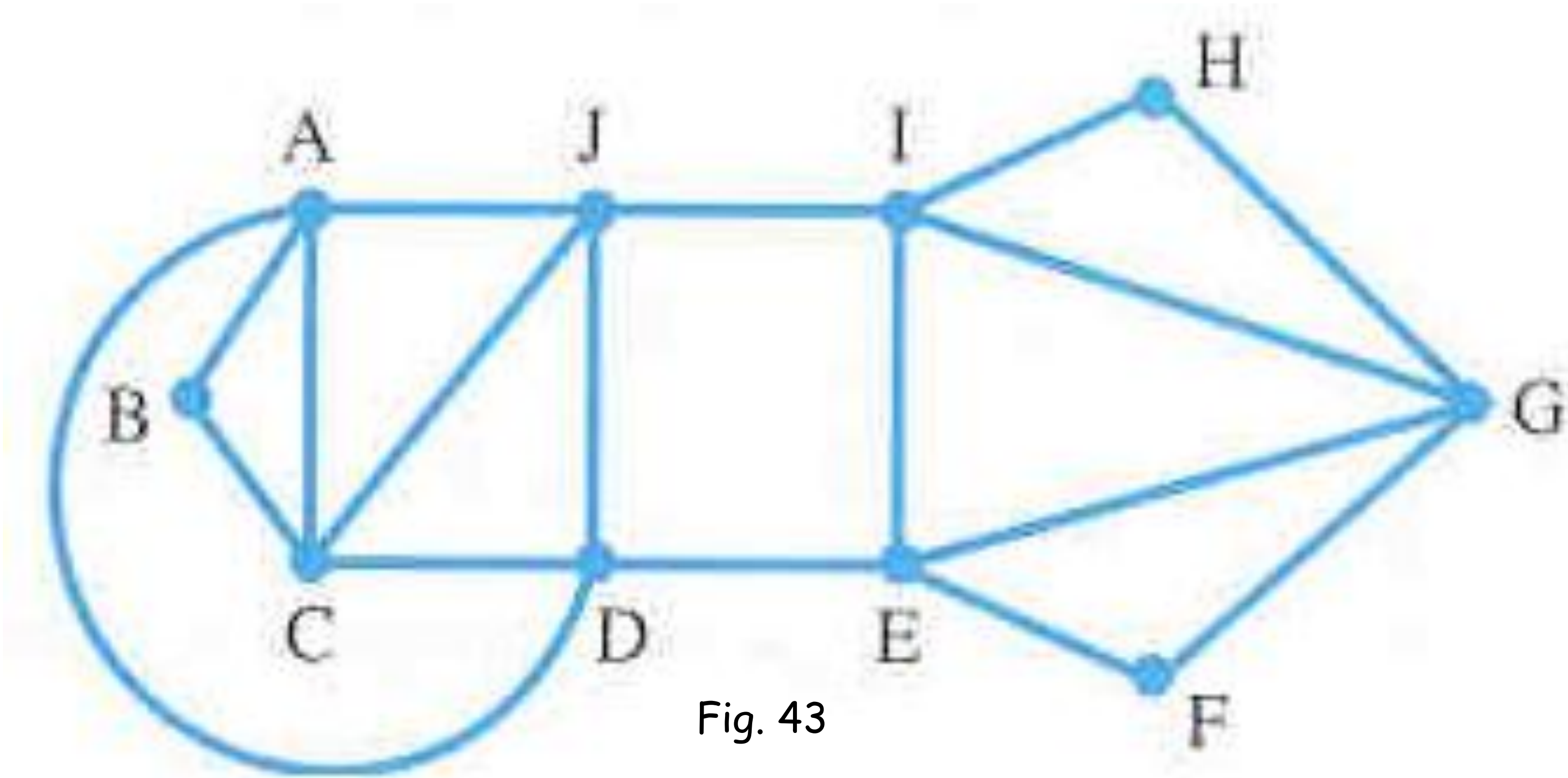
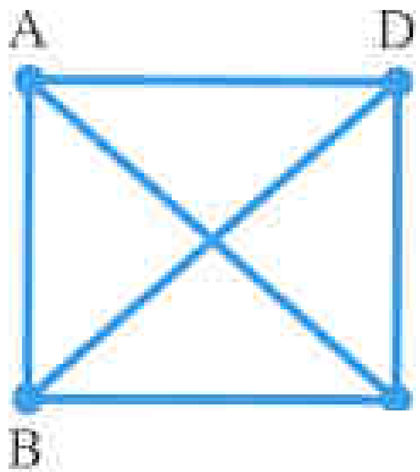


Fig. 43

44-46. In Exercises 44-46 the graph does not have an Euler circuit. For each graph find a circuit that uses as many edges as possible. (There is more than one correct answer in each case.) How many edges did you use in

Fig. 44



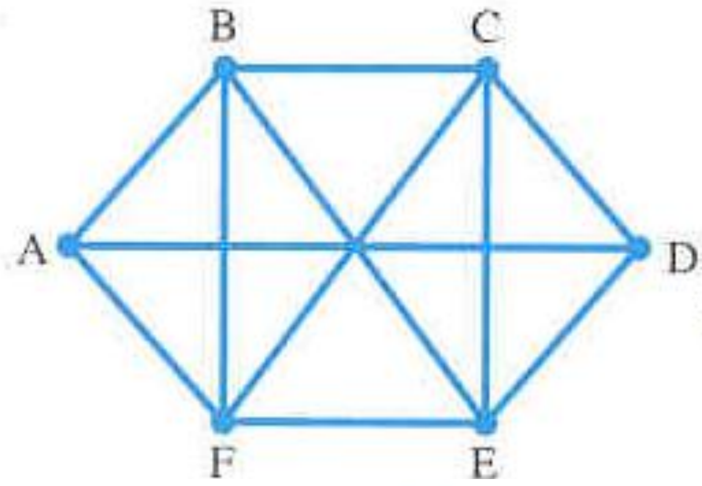


Fig. 45

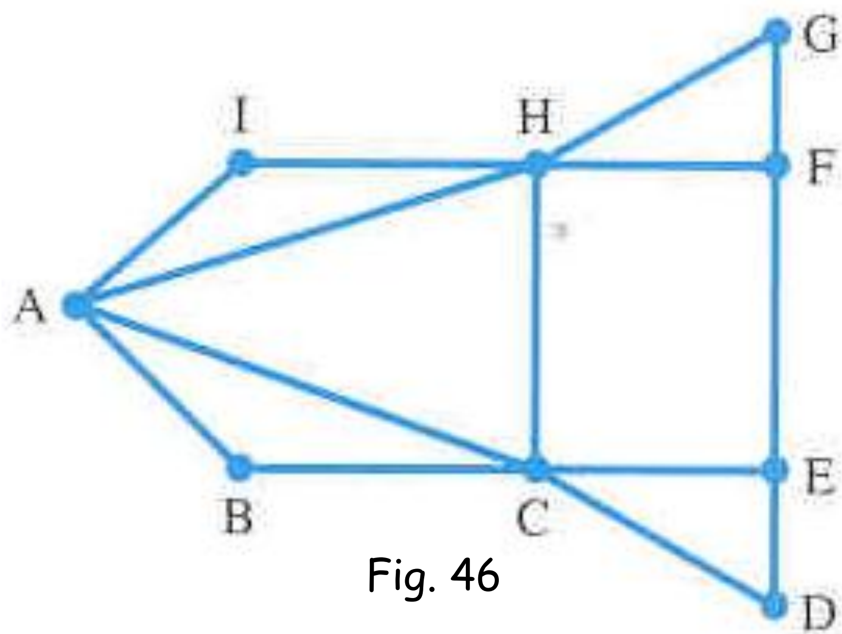


Fig. 46

47-50. In Exercises 47-50 different floor tilings are shown. The material applied between tiles is called grout. For which of these floor tilings could the grout be applied beginning and ending at the same place, without going over any section twice, and without lifting the tool? Justify answers

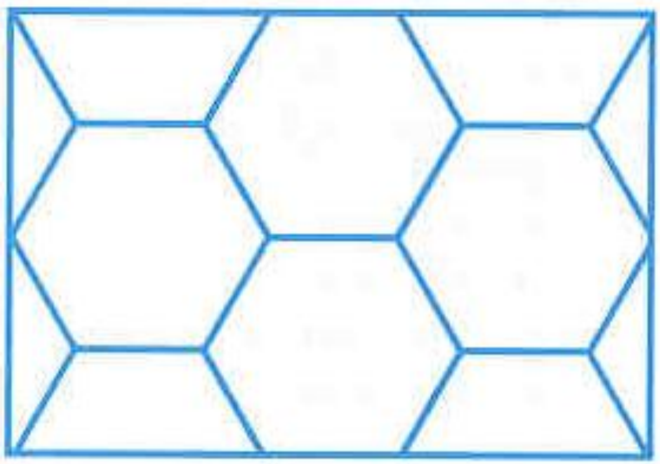


Fig. 47

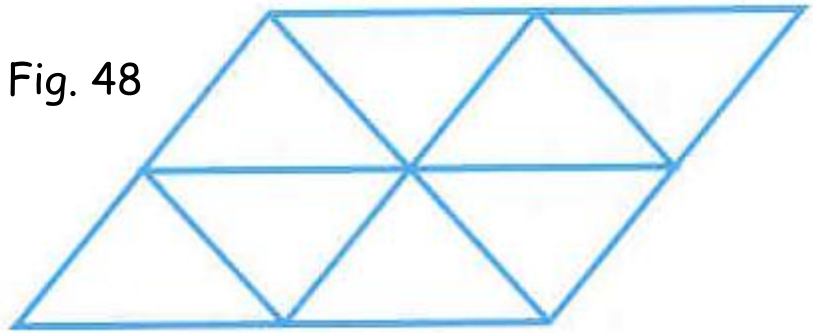


Fig. 48

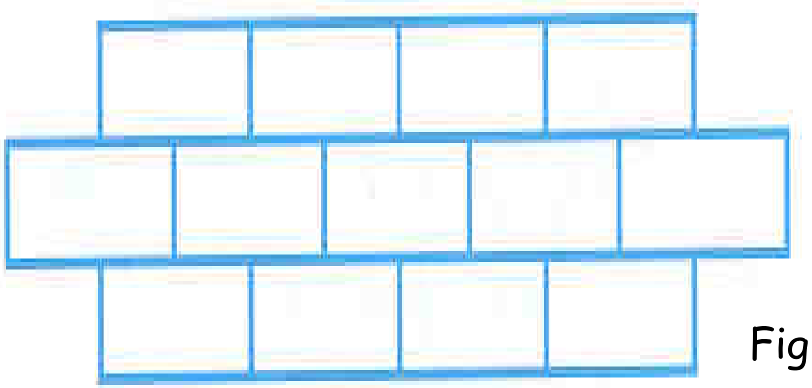


Fig. 49

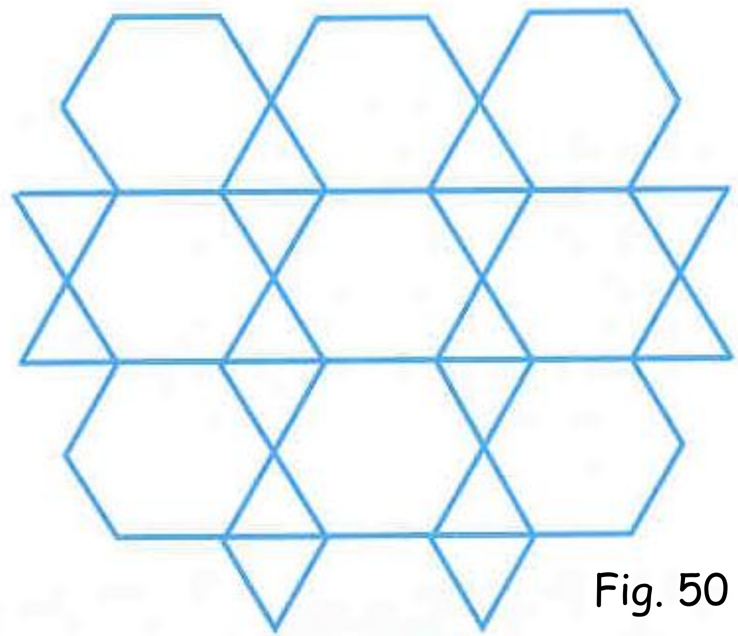


Fig. 50

51. Use Fleury's Algorithm to find an Euler circuit'

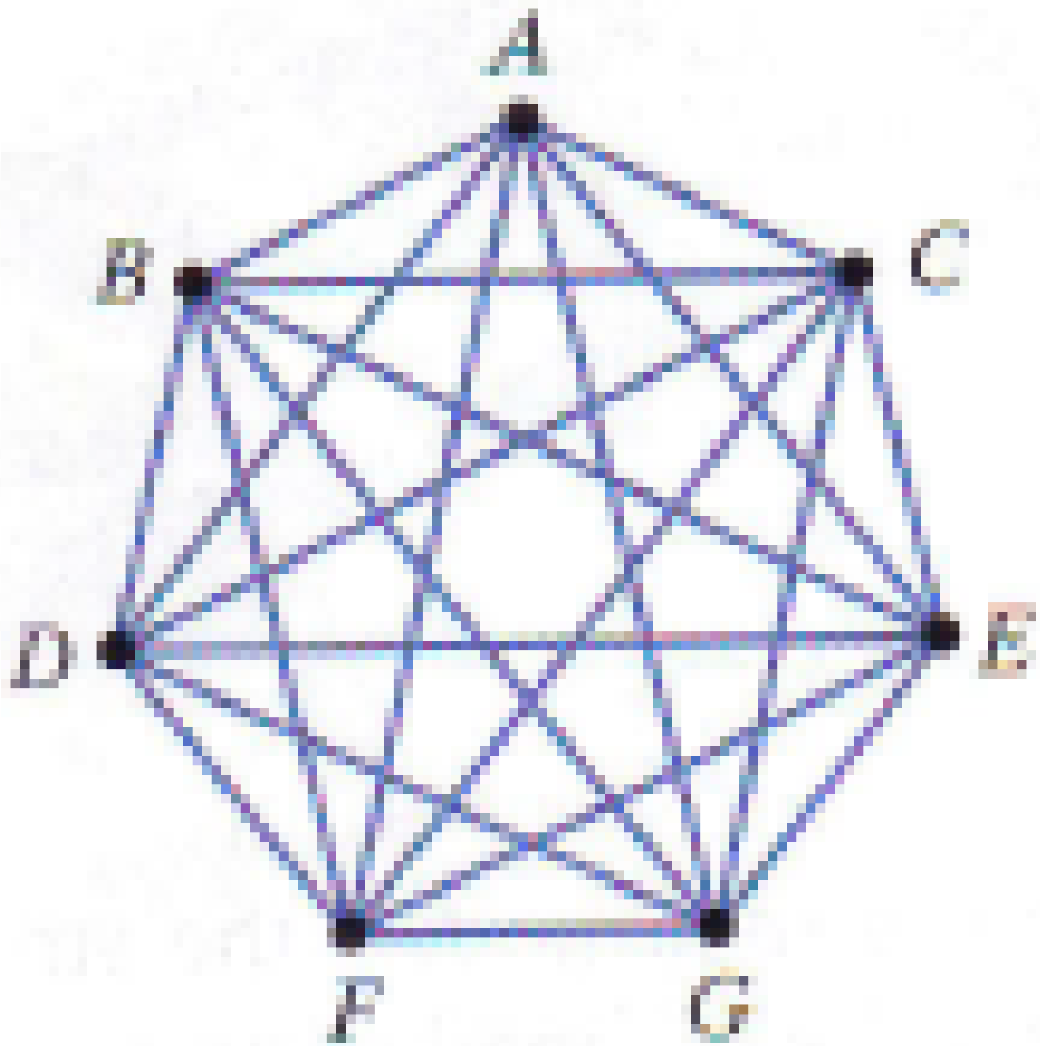
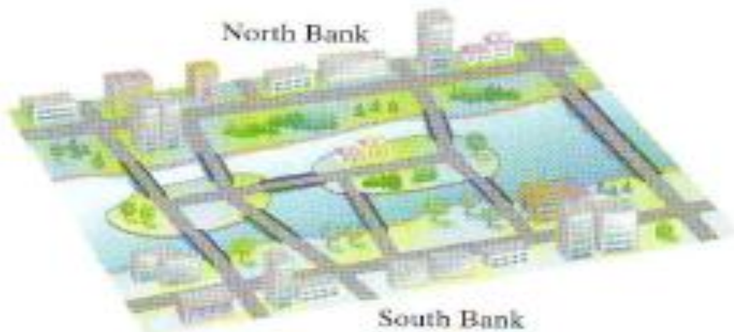
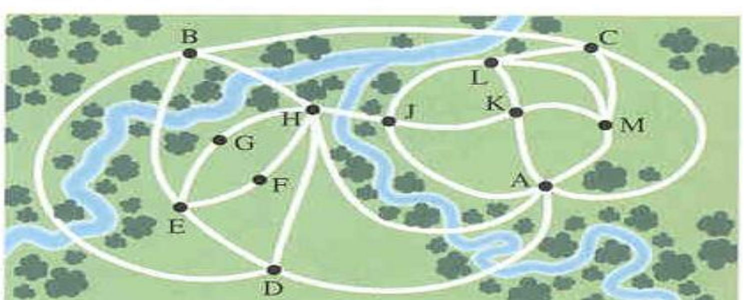
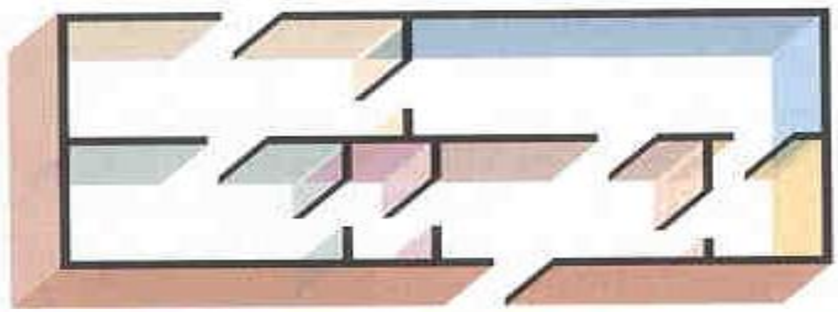


Fig. 51

4.B Applications On: Euler Paths and Circuits



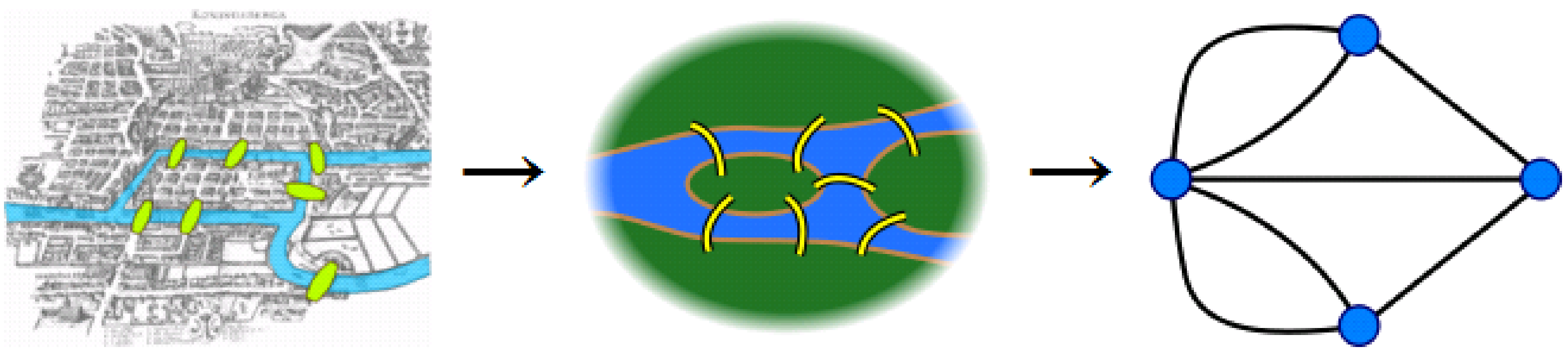




Applications On: Euler Paths and Circuits

Bridges Problem

- Seven bridge of Konigsberg problem: it is possible to start at some location at the town (Prussia), travel across all the bridges without crossing any bridge twice, and return to starting point.



- We can find a path or circuit that traverse:
  - Each street in a neighborhood exactly one.
  - Each road in a transportation network exactly one.
  - Each link in communication network exactly one.
- Euler path or circuit also applied in
  - Layout of circuits.
  - Network multitasking.
  - Molecular biology (DNA sequencing)

Check Point

The layout of a city with land masses and bridges is shown

a. Draw a graph that models the layout of the city. Use vertices to represent the land masses and edges to represent the bridges.

b. Use the graph to determine if the city residents would be able to walk across all of the bridges without crossing the same bridge twice.

c. If such a walk is possible, show the path on your graph in part (a). Then trace this route on the city map in a manner that is clear to the city's residents

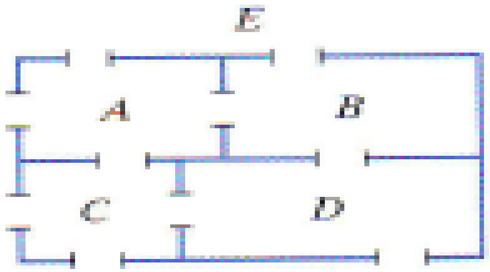
Applications On: Euler Paths and Circuits

Floors Problem

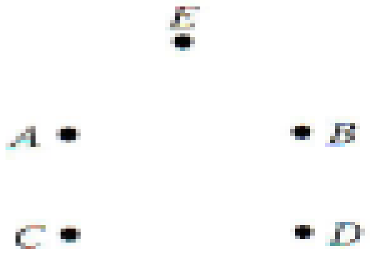
A, B, C, and D represent the rooms and, E represents the outside of the house. The edges represent the connecting doors.

a. Is it possible to find a path that uses each door exactly once?

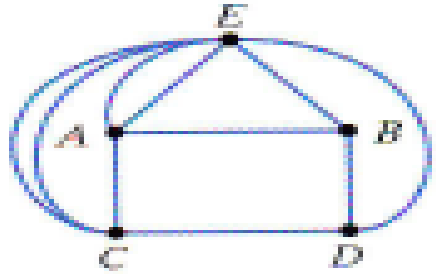
b. If it is possible, use trial and error to show such a path on the graph in Figure (c) and on the floor plan in Figure (a).



(a)



(b)



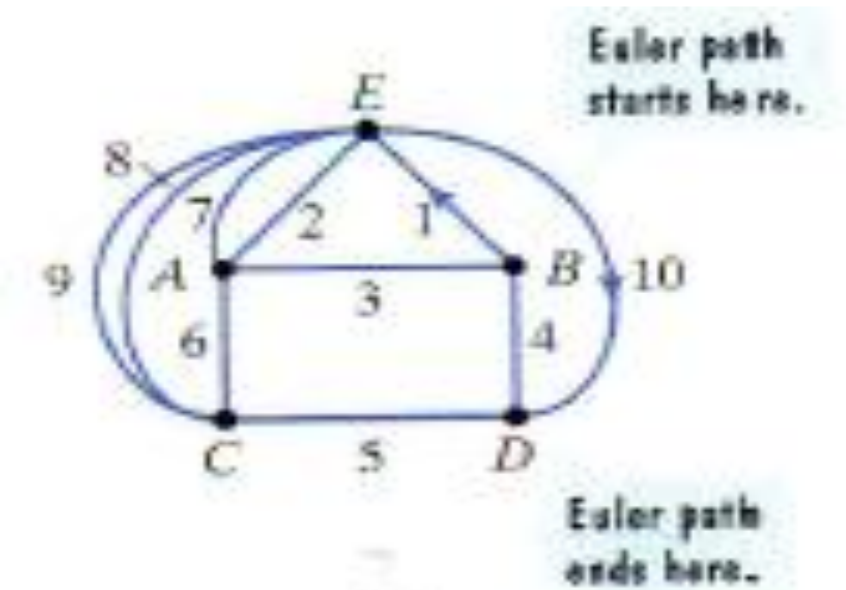
(c)

A path that uses each door (or edge) exactly once means that we are looking for an Euler path or an Euler circuit if on the graph in Figure (c) Figure indicates that there are exactly two odd vertices namely B and D. By Euler's Theorem, the graph has at least one Euler path, but no Euler circuit. It is possible to find a path that uses each door exactly once. It is not possible to begin and end the walk in the same Place.

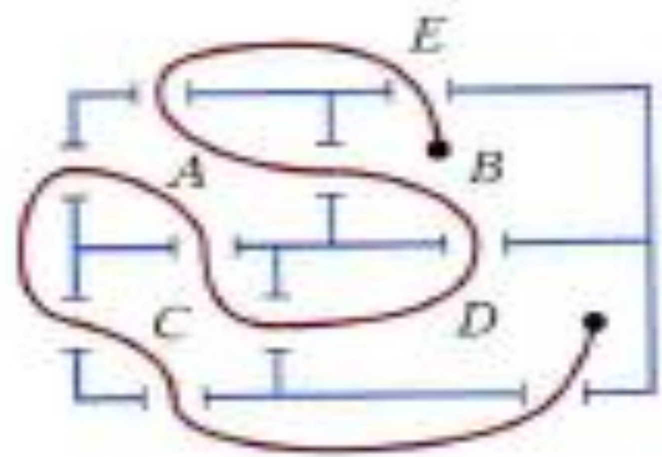
Applications On: Euler Paths and Circuits

Floor Plan Problem

- Euler's Theorem tells us that a possible Euler path must start at one of the odd vertices and end at the other one. We will use trial and error to determine an Euler path, starting at vertex B (room B in the floor plan) and ending at vertex D (room D in the floor plan). Fig. (a) shows an Euler path on the graph. Fig. (b) translates the path into a walk through the rooms.



(a)



(b)

Check Point

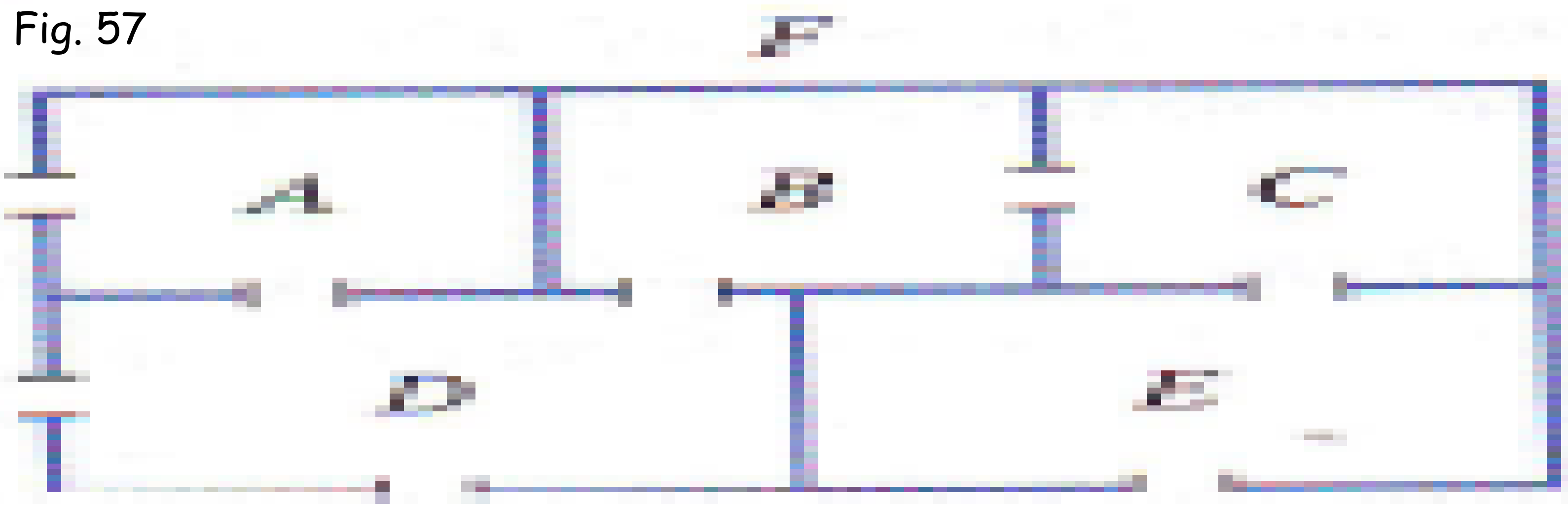
A floor plan is shown:

a. Draw a graph that models the connecting relationships in the given floor plan. (Use vertices to represent the rooms and the outside, and edges to represent the connecting doors.

b. Use your graph to determine if it is possible to find a path that uses each door only once.

c. If such a path is possible, show it on your graph in part (a.) Then trace this route on the floor plan in a manner that is clear to a person strolling through the house.

Fig. 57



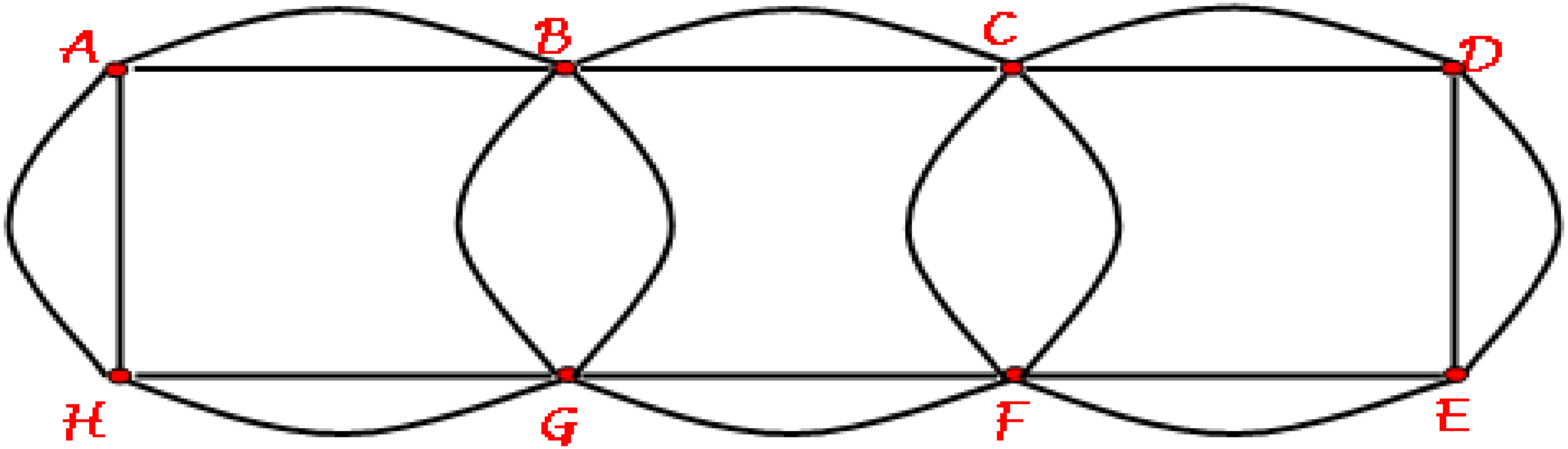
Applications On: Euler Paths and Circuits

Neighborhood Problem

A mail carrier needs to deliver mail to each house in the three-block neighborhood shown. He plans to park at one of the street intersections and walk to deliver mail. All streets have houses on both sides. This means that the mail carrier must walk down every street twice, delivering mail to each side separately.

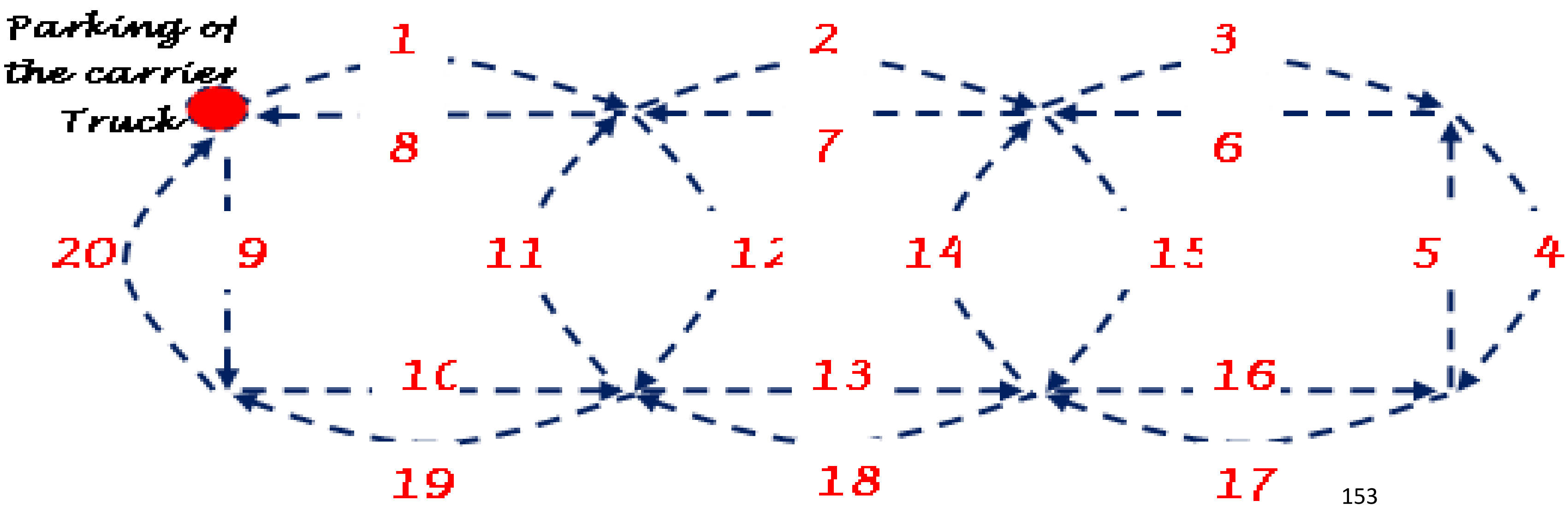
a. Draw a graph that models the streets of the neighborhood walked by the security guard.

b. Determine whether the residents in the neighborhood will be able to establish a route for the security guard so that each street is walked exactly once. If this is possible, use your map to show where the guard should begin the walk.



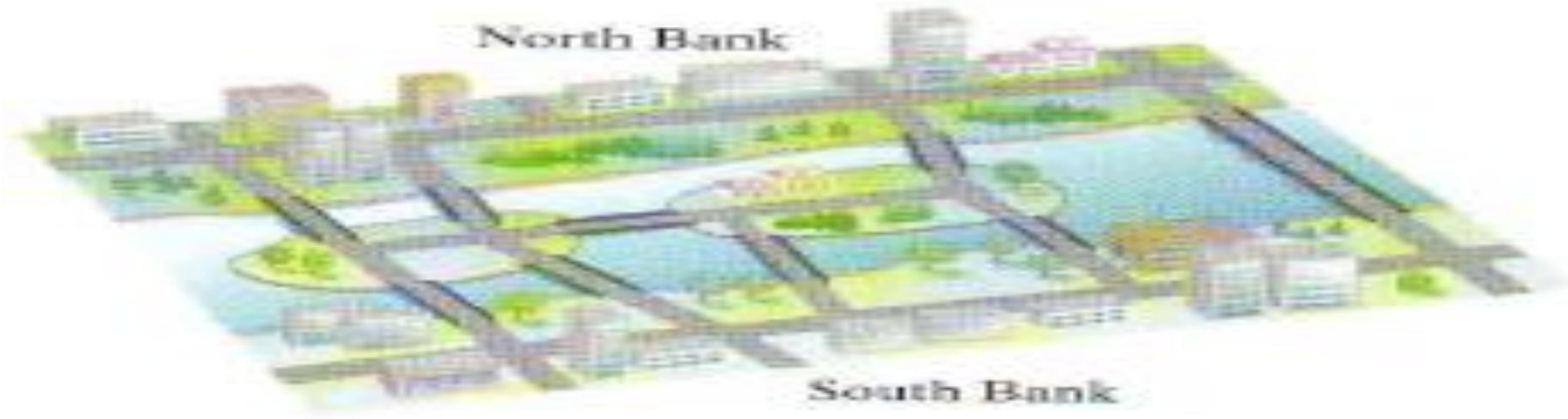
Since the graph shown above has :

The Degrees of all the vertices are Even, hence the carrier can park at an intersection, deliver mail to each house without retracing the side of any street, and return to the parked truck.



Exercise Set 4.B

1. The layout of a city with land masses and bridges is shown below.



a. Draw a graph that models the layout of the city. Use vertices to represent the land masses and edges to represent the bridges.

b. Use the graph to determine if the city residents would be able to walk across all of the bridges without crossing the same bridge twice.

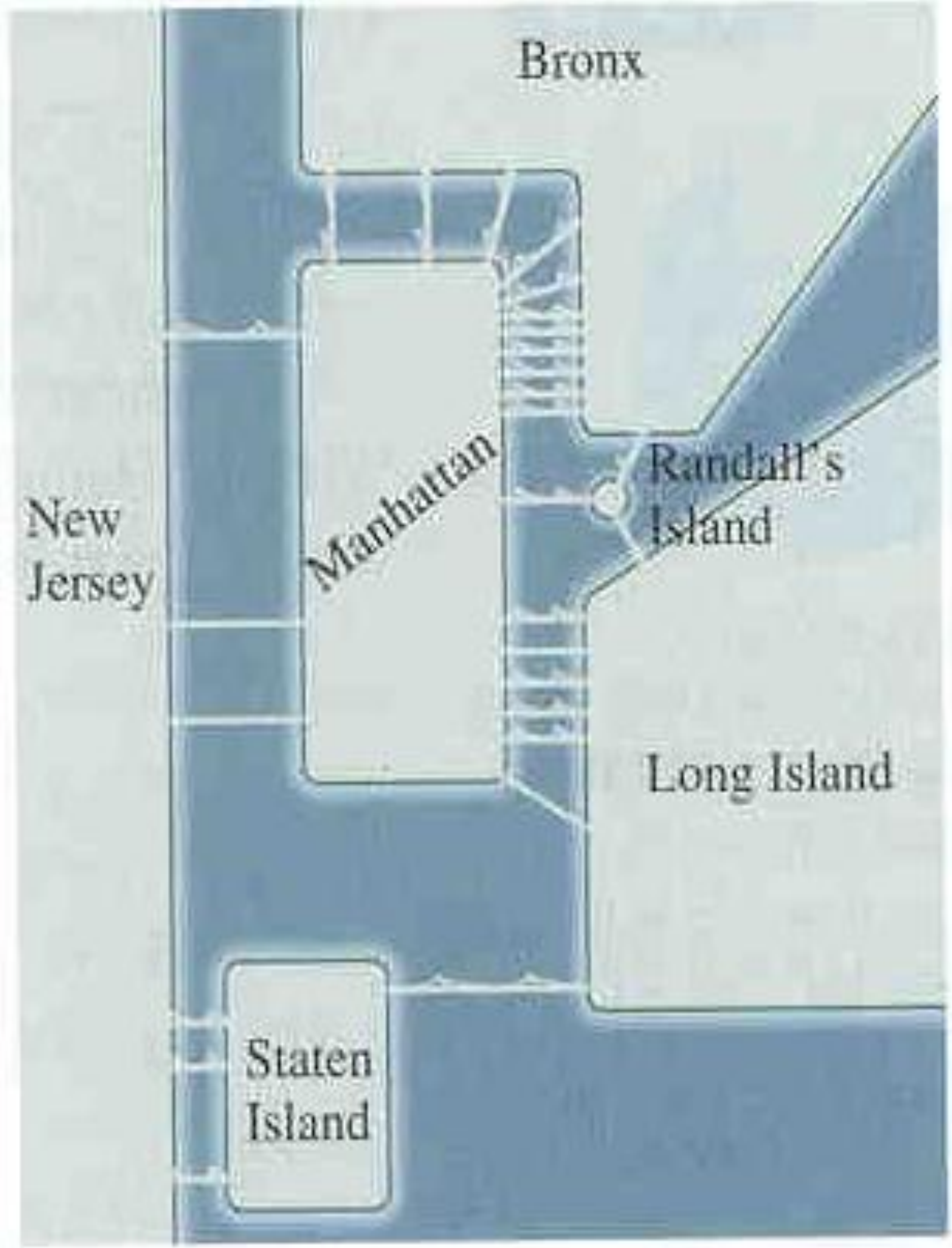
c. If such a walk is possible, show the path on your graph in part (a). Then trace this route on the city map in a manner that is clear to the city's residents

2. The figure shows a map of a portion of New york City with the bridge and tunnel connections. Use a graph to determine if it is possible to visit Manhattan, Long Island, Staten Island, and New Jersey using each bridge or tunnel only once.

3. The accompanying schematic map shows a portion of the New York City area, including tunnels and bridges.

a. Is it possible to take a drive around the New York City area using each tunnel and bridge exactly once, beginning and ending in the same place? Justify your answer.

b. Is it possible to take a drive around the New York City area using each tunnel and bridge exactly once, beginning and ending in different places? If so, where must your drive begin and end? Justify your answer.



4-7. In Exercises 4-7, the floor plan of a building is shown. For which of these is it possible to start outside, walk through each door exactly once, and end up back outside? Justify each answer. (Hint: Think of the rooms and "outside" as the vertices of a graph, and the doors as the edges of the graph.)

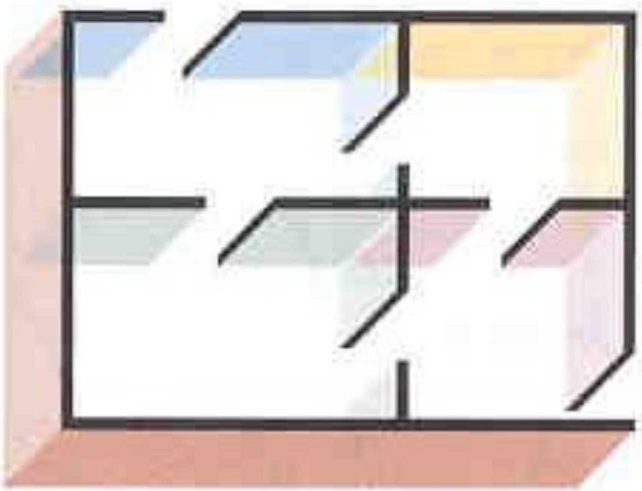


Fig. 59

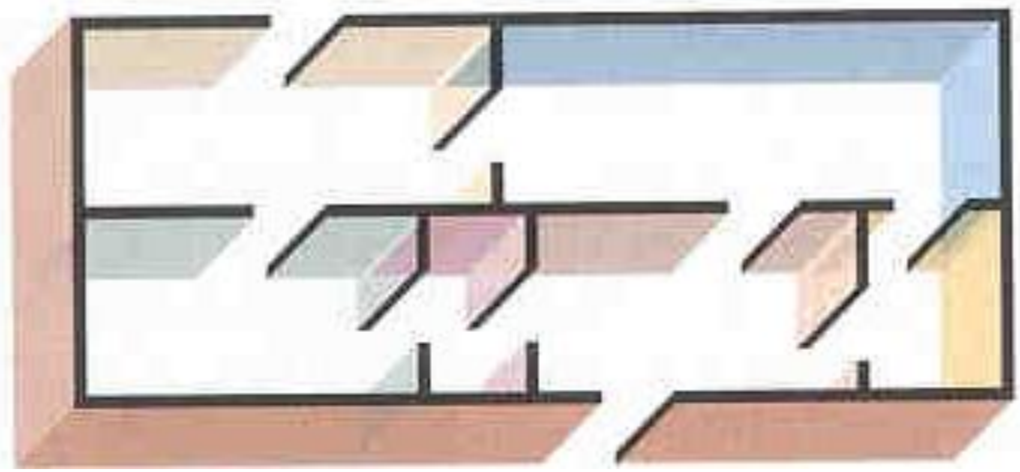


Fig. 60

Fig. 58

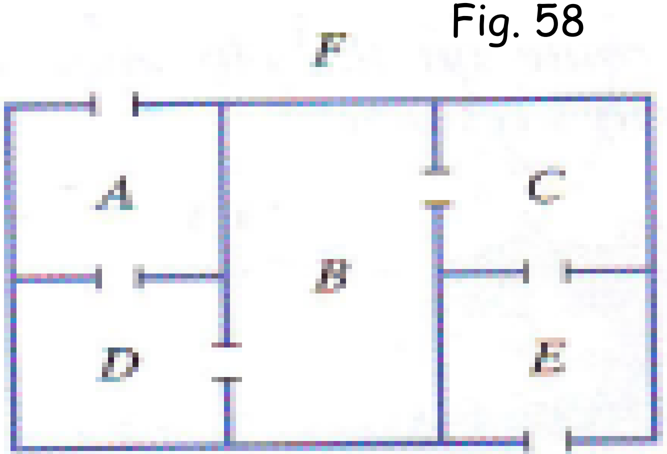
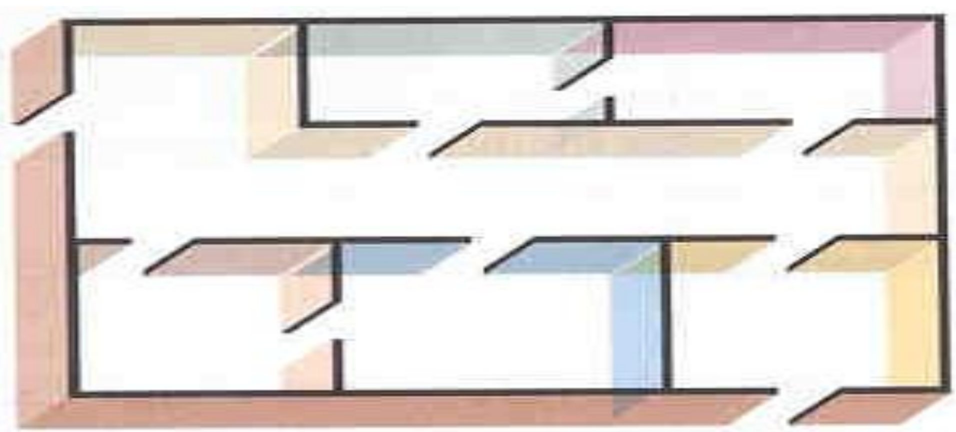


Fig. 61



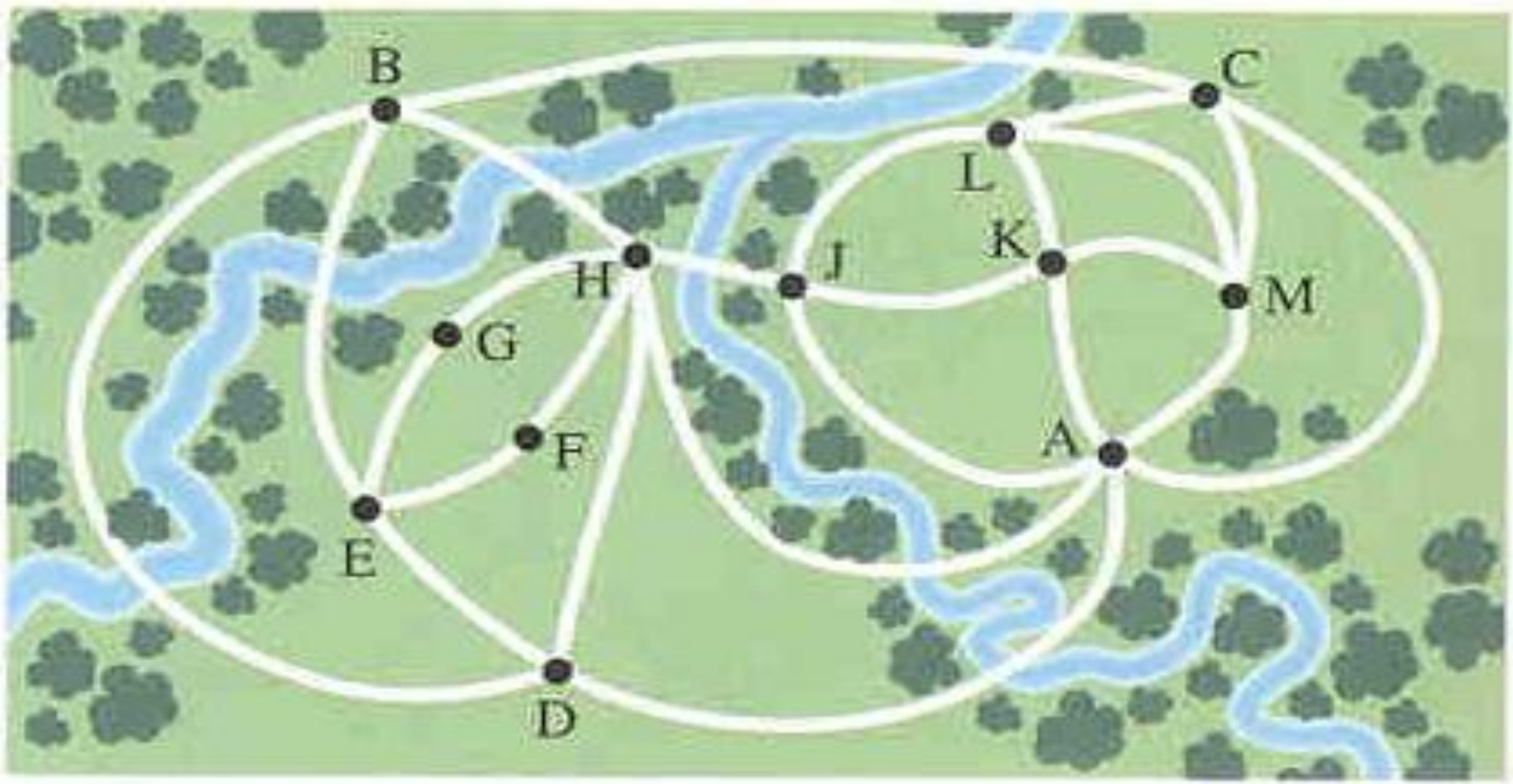
8. A security guard needs to walk the streets of the neighborhood shown, walking each street once.

a. Draw a graph that models the streets of the neighborhood walked by the security guard.

b. Determine whether the residents in the neighborhood will be able to establish a route for the security guard so that each street is walked exactly once. If this is possible, use your map to show where the guard should begin the walk.

9. A police car would like to patrol each street in the neighborhood shown on the map exactly once. Use numbering system from 1 through 31, one number per street, to show the police how this can be done'

10. The map shows the roads on which parking is permitted at a national monument. This is a pay and display facility. A security guard has the task of periodically checking that all parked vehicles have a valid parking ticket displayed. He is based at the central complex, labeled A. Is there a route *that he* can take to walk along each of the roads exactly once, beginning and ending at A? If so, use Fleury's algorithm to find such a route



Summary of: Euler's Theorem

Number of Odd Vertices	Euler Paths	Euler Circuits
None (all even)	at least one	at least one
exactly two	at least one	none
more than two	none	none

4.C Hamilton Paths & Circuits

Hamiltonian Paths and Circuits

- Discovered by Irish Mathematician Sir William Rowan Hamilton (1857)

- A Hamiltonian circuit HC : in a graph G is a simple circuit that passes through every vertex in G exactly once.

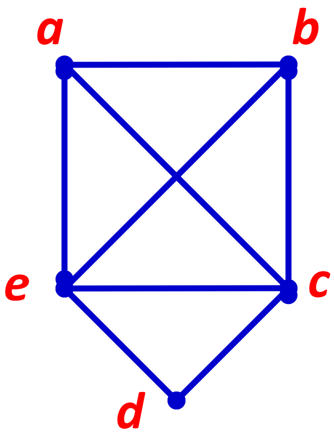
- A Hamiltonian path in a graph G is a simple path that passes through every vertex in G exactly once.

- Certain Properties In Hamiltonian Circuit (Hc):
  - A graph with a vertex of degree one cannot have HC, because each vertex in HC is incident with two edges.
  - If a vertex has degree two, then both edges that are incident with this vertex must be part of any HC.
  - When a HC is being constructed and this circuit passes through a vertex, then all remaining edges incident with this vertex, other than two used in the circuit, can be removed from consideration.
  - HC cannot contain a smaller circuit within it.

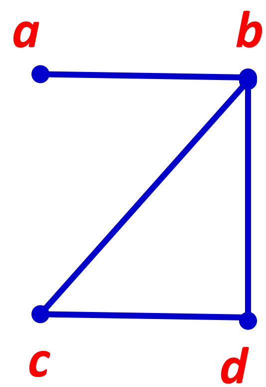
Hamiltonian Paths and Circuits

- <u>Necessary & Sufficient Conditions</u>:
  - The more edges a graph has, the more likely it is to have a HC.
  - Adding edges (but not vertices) to a graph with a HC produces a graph with the same HC.
  - DIRAC'S Theorem: If G is a simple graph with n vertices with $n \geq 3$ such that the degree of every vertex in G is at least $n/2$, then G has a HC.
  - ORE'S Theorem: If G is a simple graph with n vertices with $n \geq 3$ such that $\deg(u) + \deg(v) \geq n$ for every pair of nonadjacent vertices u and v in G, then G has a HC.

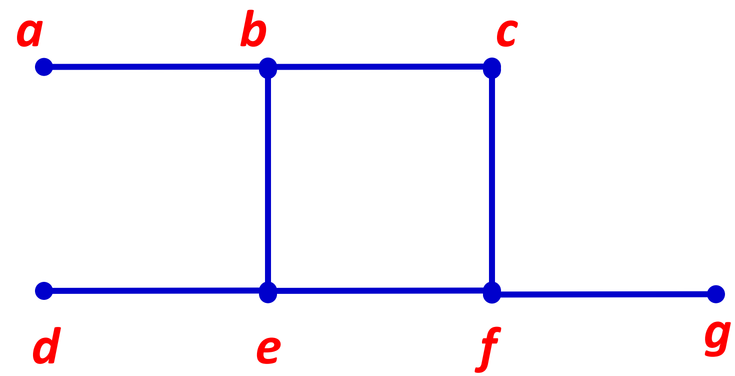
Hamiltonian Paths and Circuits Examples



G_1



G_2

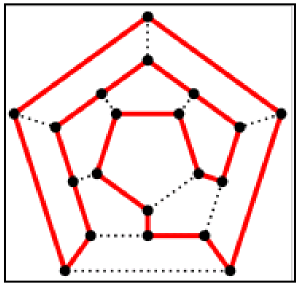


G_3

- Hamiltonian Circuit: G_1 (a, b, c, d, e, a)
- Hamiltonian Path: G_2 (a, b, c, d)
- G_3: Neither Hamiltonian path nor Hamiltonian circuit

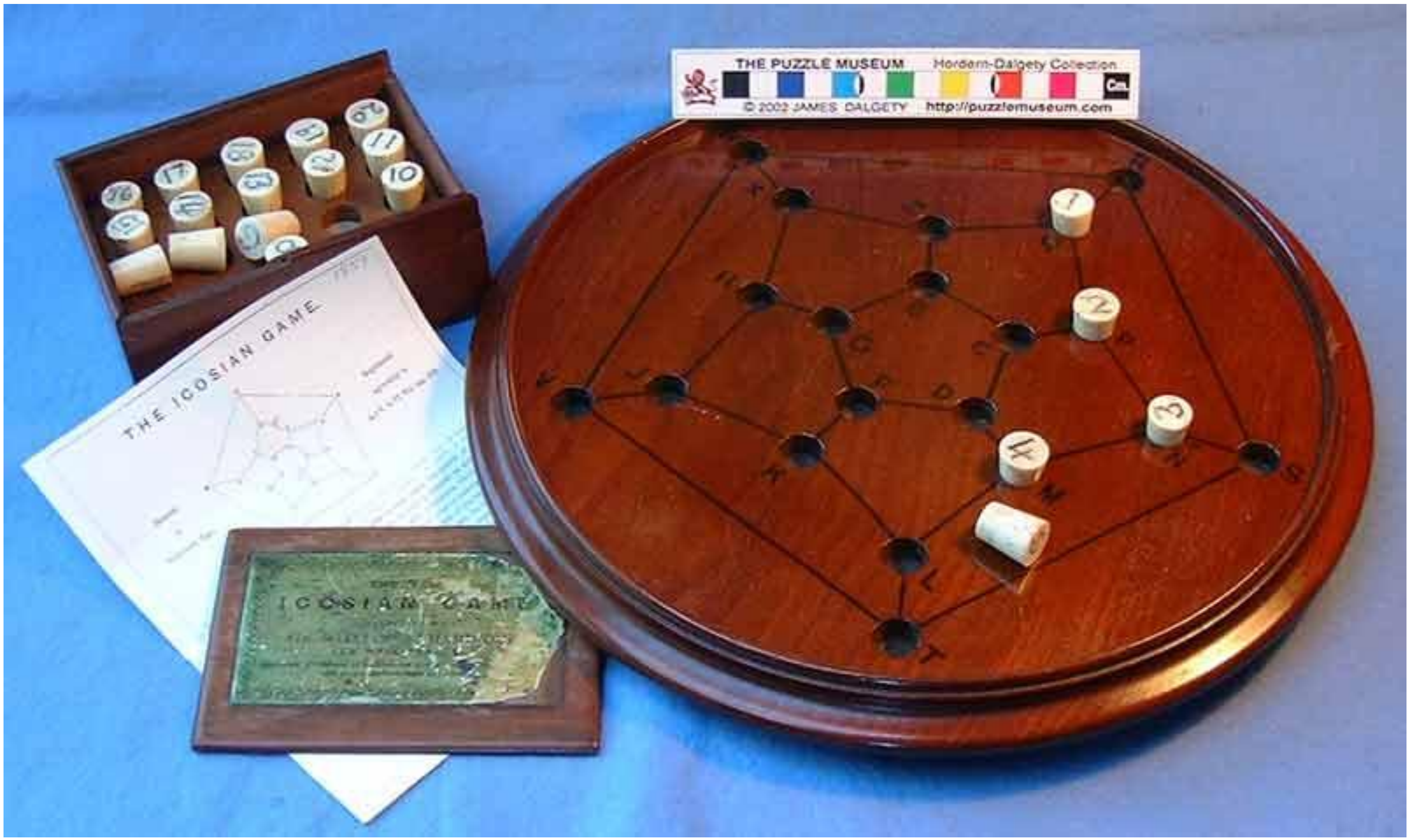
Hamiltonian Paths and Circuits Applications

- Icosian Games
  - a mathematical game invented in 1857 by William Rowan Hamilton. The game's object is finding a Hamiltonian cycle along the edges of a dodecahedron such that every vertex is visited a single time, no edge is visited twice, and the ending point is the same as the starting point. The puzzle was distributed commercially as a pegboard with holes at the nodes of the dodecahedral graph and was subsequently marketed in Europe in many forms.

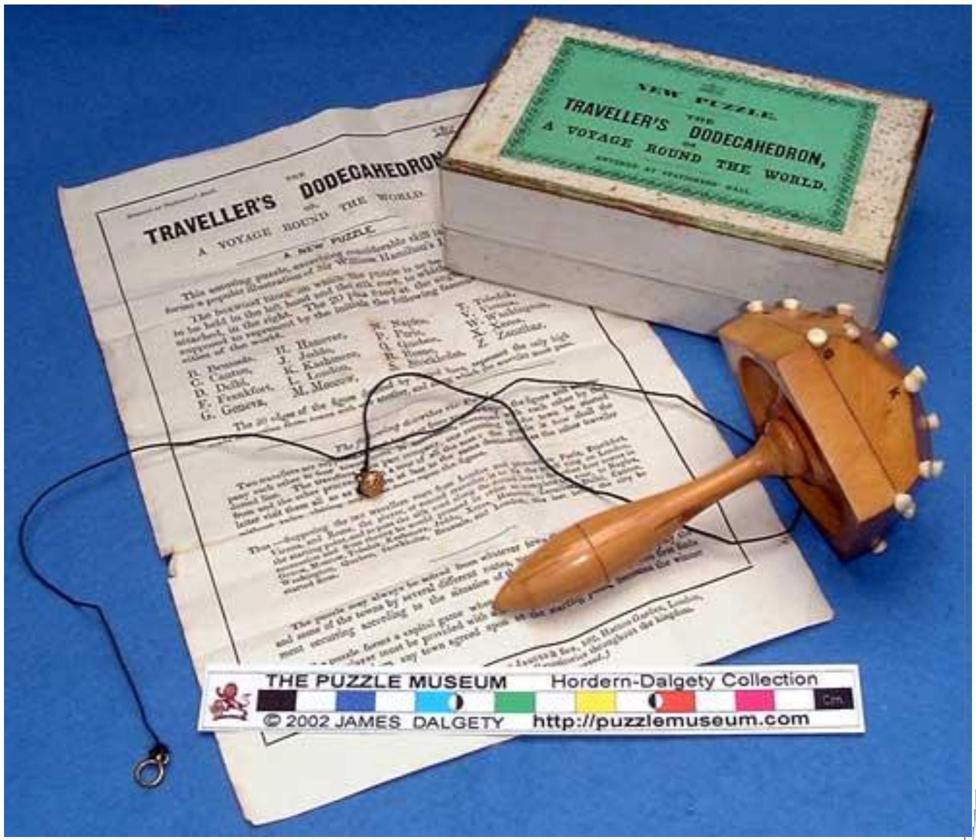


- We can find a path or circuit that visits:
  - Each road intersection in a city exactly once.
  - Each place pipelines intersect in a utility grid.
  - Each node in a communications network exactly once.
- Euler path or circuit also applied in
  - Traveling salesman problem which asks for the shortest route a traveling salesman should take to visit a set of cities.
  - Gray Code: labeling the arcs of a circle such that adjacent arcs are labeled with bit strings that differ in exactly one bit.

The Icosian Game



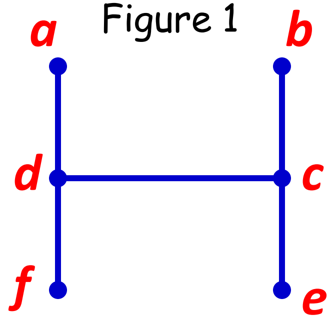
(Or the travel version!)



Check Point

Which of the following graphs have an Hamiltonian Circuit? Of those that do not, which have an Hamiltonian path?

Figure 1



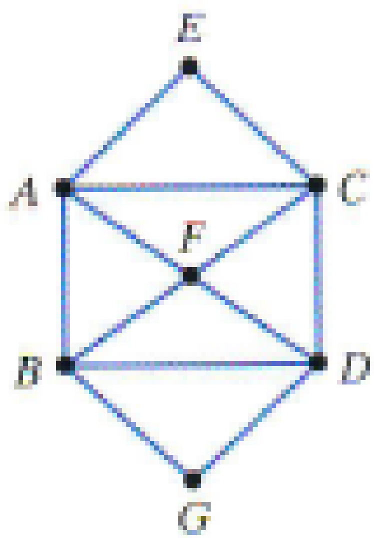


Figure 2

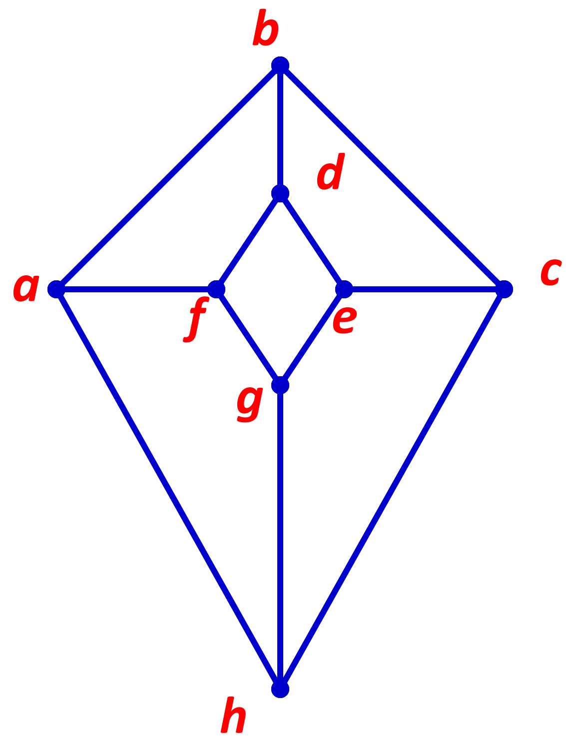


Figure 3

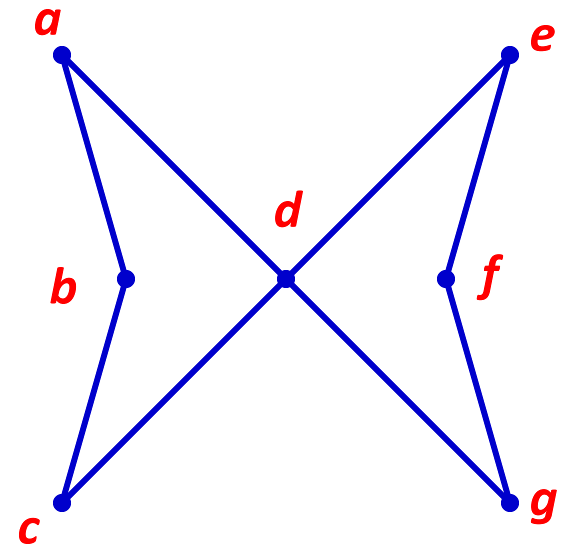
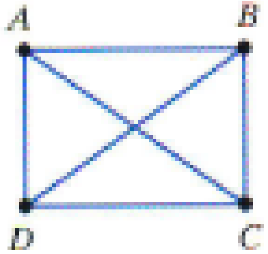


Figure 4

The Number of Hamilton Circuits in a Complete Graph

The graph in RHS has an edge between each pair of its four vertices. Thus, the graph is complete and has a Hamilton circuit. Actually, it has a complete repertoire/list/collection/stock of Hamilton circuits.



For example, one Hamilton circuit is A, B, C, D,A. Any two circuits that pass through the same vertices in the same order will be considered to be the same. For example, here are four different sequences of letters that produce the same Hamilton circuit on the given graph.

A, B, C, D, A and B, C, D, A, B and C, D, A, B, C and D , A , B , C , D.

The Hamilton circuit passing through A, B, C, and D clockwise along the four outside edges can be written in four ways.

In order to avoid this duplication, in forming a Hamilton circuit, we can always assume that it begins at A.

If a graph has **n** vertices, once we list vertex, A, there are n - 1 remaining letters. The number of Hamilton circuits depends upon the number of permutations of the "n – 1" letters, and it equals to **(n – 1)!**.

The Number of Hamilton Circuits in a Complete Graph

Determine the number of Hamilton circuits in a complete graph with

a. four vertices. b. five vertices. c. eight vertices

a. A complete graph with four vertices has (4 - 1) ! = 3! = 3.2.1 = 6 Hamilton circuits. These are the six circuits that we listed at the previous slide.

b. A complete graph with five vertices has (5 - 1)! =4! = 4 . 3 . 2 . 1 = 24 Hamilton circuits.

c. A complete graph with eight vertices has (8 - 1)! = 7! = 7.6.5. 4.3.2.1 = 5040 Hamilton circuits.

As the number of vertices in a complete graph increases, notice how quickly the number of Hamilton circuits increases.

Check Point

Determine the number of Hamilton circuits in a complete graph with

a. Three vertices.

b. Six vertices.

c. Ten vertices

Comment on your answers.

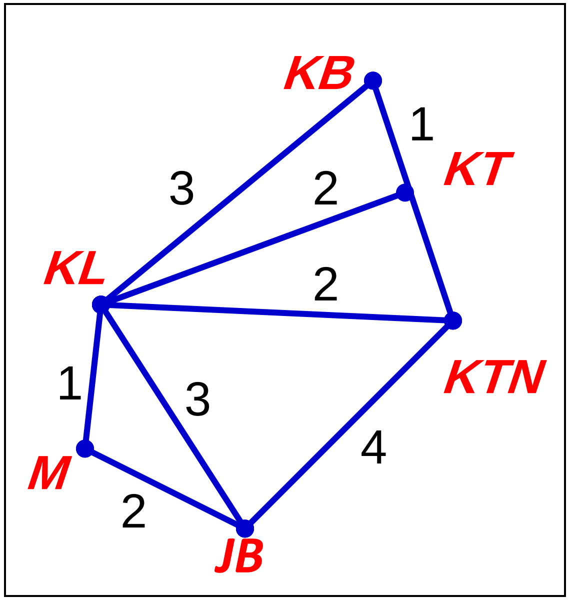
Weighted Graphs

- Understand the weighted graph.
- Understand and solve traveling salesman problem.

Weighted Graphs

- Graphs that have a number assigned to each edge are called weighted graphs.

- Problems which modeled using weighted graphs
  - Airline system (vertices: cities, edges: flight distances/ times/fares)
  - Computer network (vertices: computers, edges: communication costs/response times/ distances)



- Types of problems need to solve:
  - Shortest path/ path of least length:
    - Smallest flight time, cheapest fare, fastest response time, shortest distance.

  - A circuit with shortest total length that visits every vertex of a complete graph exactly once:
    - Traveling salesman problem.

Exercise Set 4.C

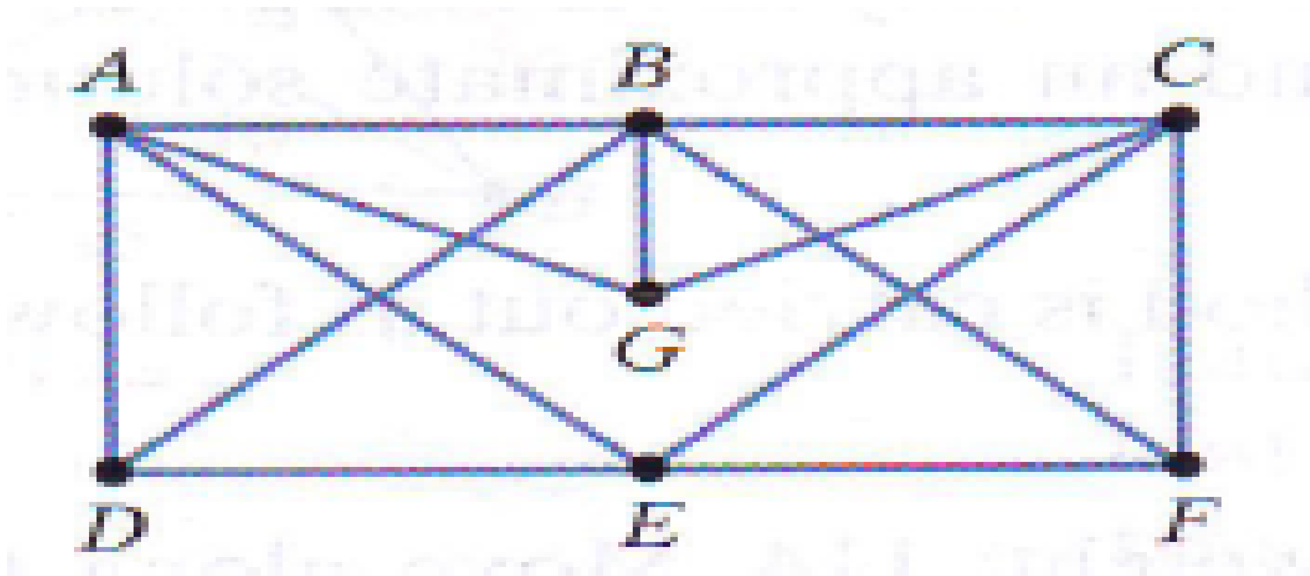
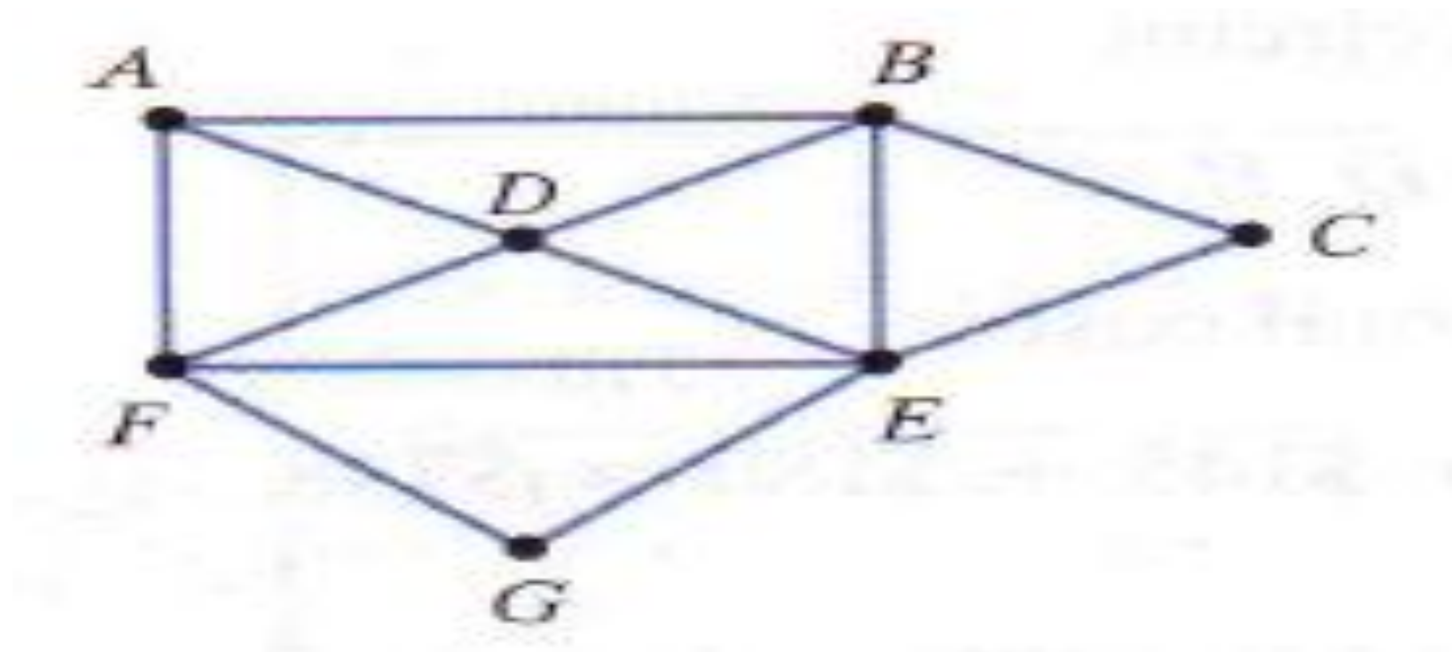


Figure 1

1-4 For the graph shown in Fig. 1

1. Find a Hamilton path that begins at,4 and ends at B.
2. Find a Hamilton path that begins at G and ends at E.
3. Find a Hamilton circuit that begins as A, B, . . .
4. Find a Hamilton circuit that begins as A, G, ...

Figure 2



5-8 For the graph shown in Fig. 2

5. Find a Hamilton path that begins at,4 and ends at D.

6. Find a Hamilton path that begins at A and ends at G.

7. Find a Hamilton circuit that begins at.4 and ends with the pair of vertices D , A.

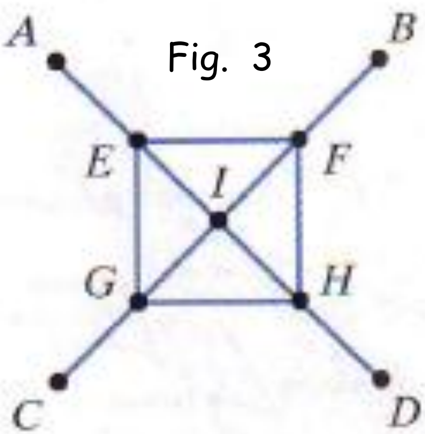
8. Find a Hamilton circuit that begins at F and ends with the pair of vertices D, F.

9-14 For the graphs in Figures 9 -14,

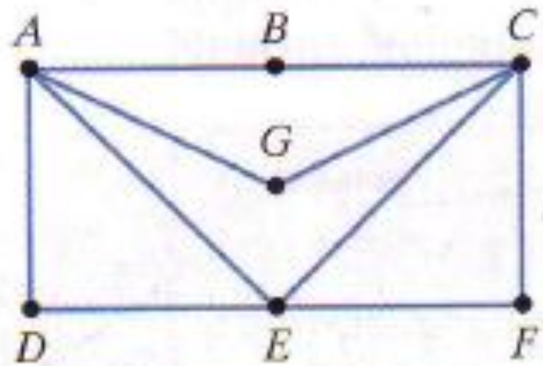
a, Determine if the graph must have Hamilton circuits, Explain your answer.

b. If the graph must have Hamilton circuits, determine the number of such circuits.

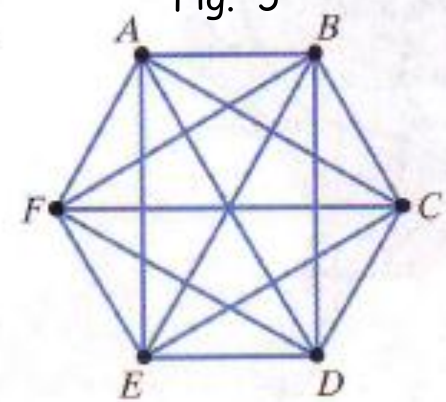
9. Fig. 3



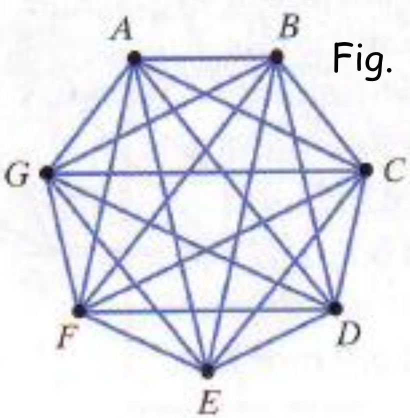
10. Fig. 4



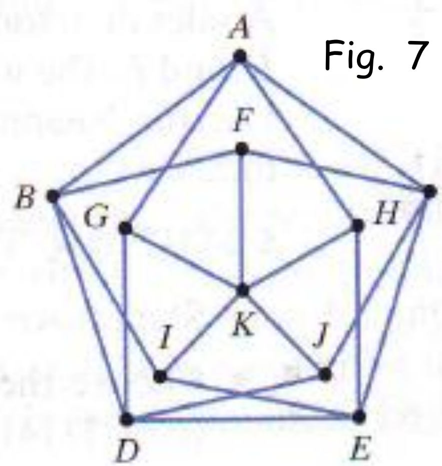
11. Fig. 5



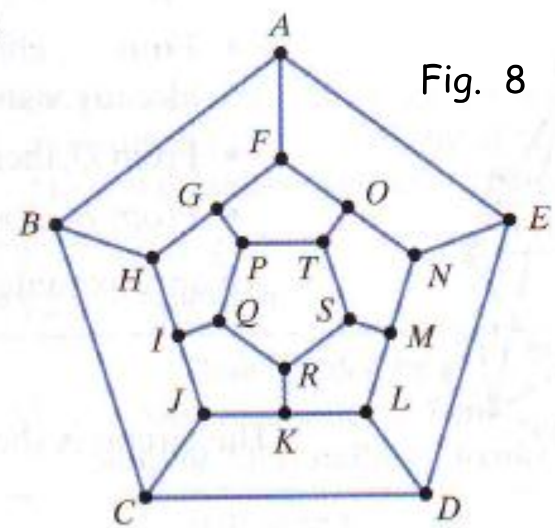
12. Fig. 6



13. Fig. 7



14. Fig. 8



In Exercises 15-18, determine the number of Hamilton circuit in a complete graph with the given number of vertices:

15. 3 16. 4 17. 12 18. 13

In Exercises 19-24, use the complete weighted graph shown in fig. 9 to:

19. Find the weight of edge CE.

20. Find the weight of edge BD.

21. Find the total weight of the Hamilton circuit A, B, C, E, D, A.

22. Find the total weight of the Hamilton circuit A, B, D, C, E, A.

23. Find the total weight of the Hamilton circuit A, B, D, E, C, A.

24. Find the total weight of the Hamilton circuit A, B, E, C, D, A.

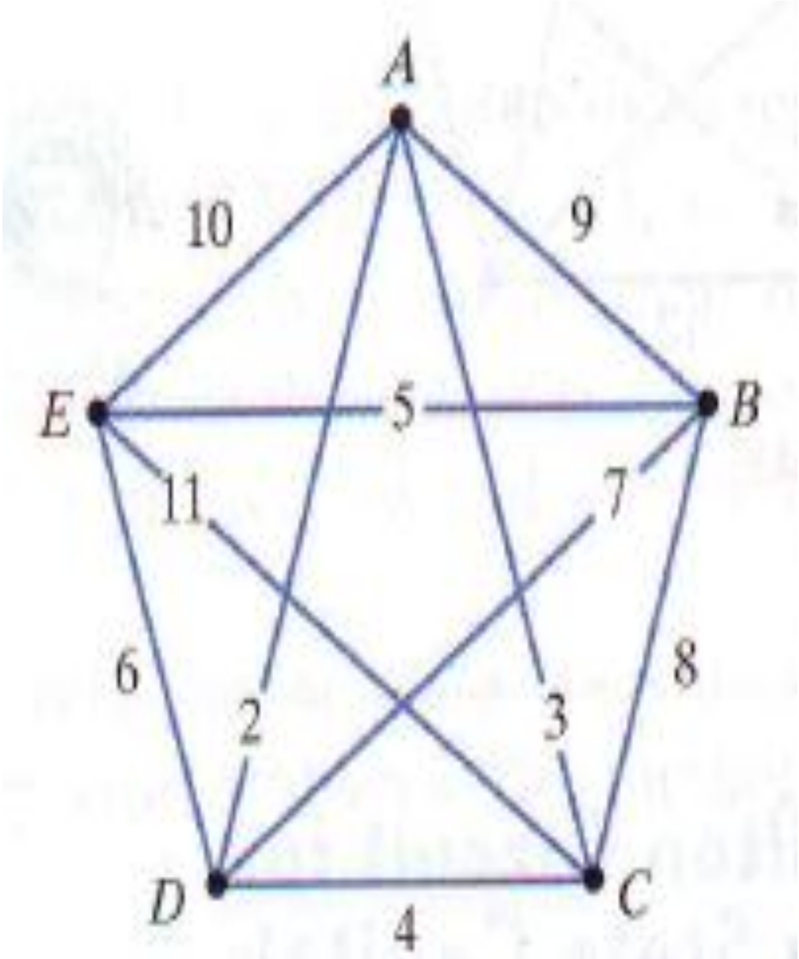


Fig. 9

In Exercises 25-34, use the complete weighted graph shown in fig. 10 to:

25. Find the total weight of the Hamilton circuit A, B, C, D, A.

26. Find the total weight of the Hamilton circuit A, B, D, C, A.

27. Find the total weight of the Hamilton circuit A, C, B, D, A.

28. Find the total weight of the Hamilton circuit A, C, D, B, A.

29. Find the total weight of the Hamilton circuit A, D, B, C, A.

30. Find the total weight of the Hamilton circuit A, D, C, B, A.

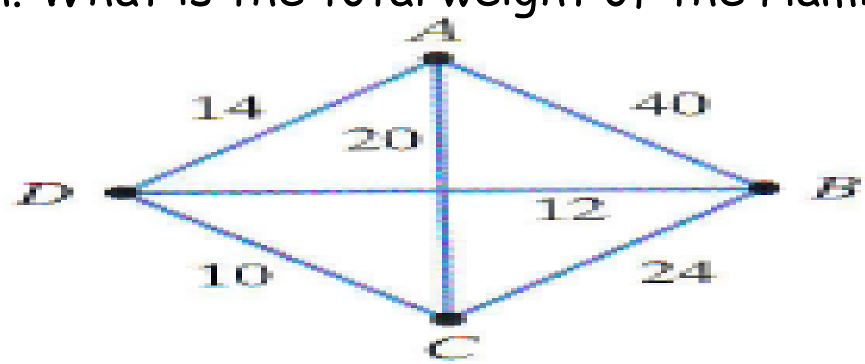
31. Use your answers from Exercises 25-30 and the Brute Force Method to find the optimal solution.

32. Use the Nearest Neighbor Method, with starting vertex A, to find an approximate solution. What is the total weight of the Hamilton circuit?

33. Use the Nearest Neighbor Method, with starting vertex B, to find an approximate solution. What is the total weight of the Hamilton circuit?

34. Use the Nearest Neighbor Method, with starting vertex C. to find an approximate solution. What is the total weight of the Hamilton circuit?

Fig. 10



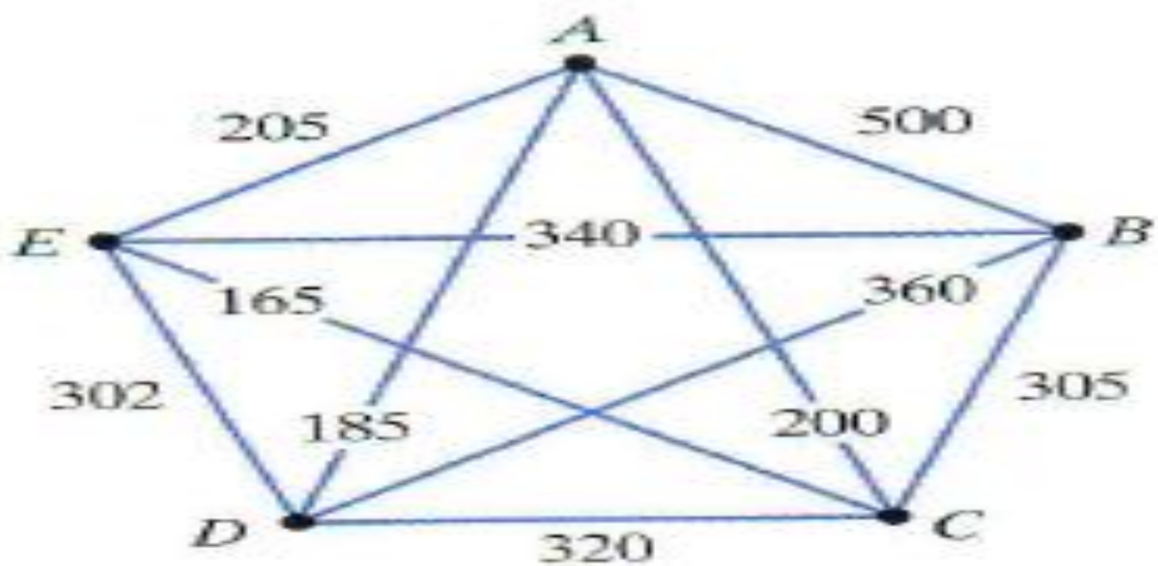
In Exercises 35 - 36, a sales director who lives in city A is required to fly to regional offices in cities B, C, D, and E. The weighted graph in figure 11 shows the one-way air fares between any two cities.

35. Use the Brute Force Method to find the optimal solution. Describe what this means for the sales director. (Hint: Because the airfares are the same in either direction' you need only compute the total cost for 12 Hamilton circuits:

A, B, C, D, E, A; A, B, D, C, E, A; A, B, E, C, D, A; A, C, B, D, E, A;
A, C, D, B, E, A; A, D, B, C, E, A; A, B, C, E, D, A; A, B, D, E, C, A;
A, B, E, D, C, A; A, C, B, E, D, A; A, C, E, B, D, A; A, D, C, B, E, A.)

36. Use the Nearest Neighbor Method, with starting vertex A, to find an approximate solution' What is the total cost for this Hamilton circuit?

Fig. 11



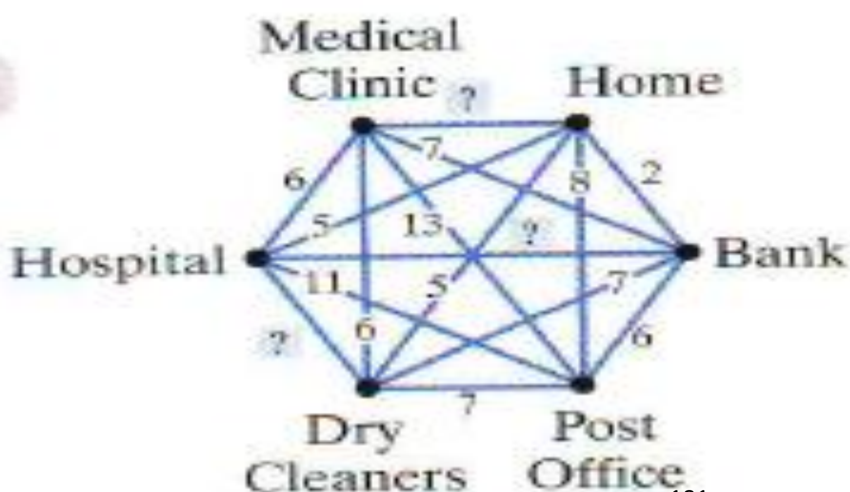
37-40. You have five errands to run around town, in no particular order. You plan to start and end at home . You must go to the post office, deposit a check at the bank, drop off dry cleaning, visit a friend at-the hospital, and get a flu shot. The map shows your home and the locations of your five errands. Each block represents one mile. Also shown is a weighted graph with distances on the appropriate edges. Your goal is to run thee errands and return home using the shortest route.

37. Use the map to fill in the three missing weights in the graph.

38. If each Hamilton circuit represents a route to run your errand, show many different routes are possible?.

39. Using the Brute Force Method, the optimal solution is Home, Bank, Post Office, Dry Cleaners, Hospital, Medical Clinic, Home. What is the total length of the shortest route?

40. Use the Nearest Neighbor Method to find an approximate solution. What is the total length of the shortest route using this solution? How does this compare with your answer to Exercise 39?



4.D Important Applications Of Hamilton's Circuit

- Weighted Graphs and the Traveling Salesperson Problem:
  - Brute Force Algorithm.
  - Nearest Neighbor Method.

- Graph Coloring

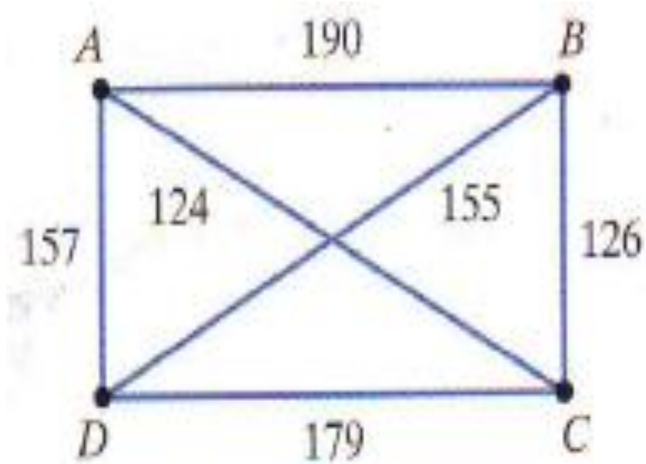
Weighted Graphs and the Traveling Salesperson Problem

Sales directors for large companies are often required to visit regional offices in a number of different cities. How can these visits be scheduled in the cheapest possible way?

For example, a sales director who lives in city A is required to fly to regional offices in cities B, C, and D. Other than starting and ending the trip in city A, there are no restrictions as to the order in which the other three cities are visited.

The one-way fares between each of the four cities are given in given Table. A graph that models this information is shown in the given Figure. The vertices represent the cities. The airfare between each pair of cities is shown as a number on the respective edge.

	A	B	C	D
A	*	$190	$124	$157
B	$190	*	$126	$155
C	$124	$126	*	$179
D	$157	$155	$179	*



Weighted Graphs and the Traveling Salesperson Problem

Brute Force Method

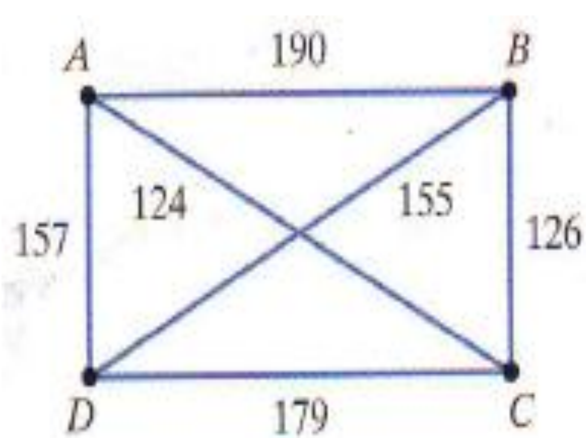
One method for finding an optimal Hamilton circuit is called the Brute Force Method.

The optimal solution is found using the following steps:

1. Model the problem with a complete, weighted graph.
2. Make a list of all possible Hamilton circuits.
3. Determine the sum of the weights of the edges for each of these Hamilton circuits.
4. The Hamilton circuit with the minimum sum of weights is the optimal solution

Weighted Graphs and the Traveling Salesperson Problem

The traveling sales person problem is the problem of finding a Hamilton circuit in a complete weighted graph for which the sum of the weights of the edges is a minimum. Such a Hamilton circuit is called the Optimal Hamilton Circuit or the Optimal Solution.



Hamilton Circuit	Sum of the Weights of the Edges	=	Total Cost
A, B, C, D, A	190 + 126 + 179 + 157	=	\$652
A, B, D, C, A	190 + 155 + 179 + 124	=	\$648
A, C, B, D, A	124 + 126 + 155 + 157	=	\$562
A, C, D, B, A	124 + 179 + 155 + 190	=	\$648
A, D, B, C, A	157 + 155 + 126 + 124	=	\$562
A, D, C, B, A	157 + 179 + 126 + 190	=	\$652

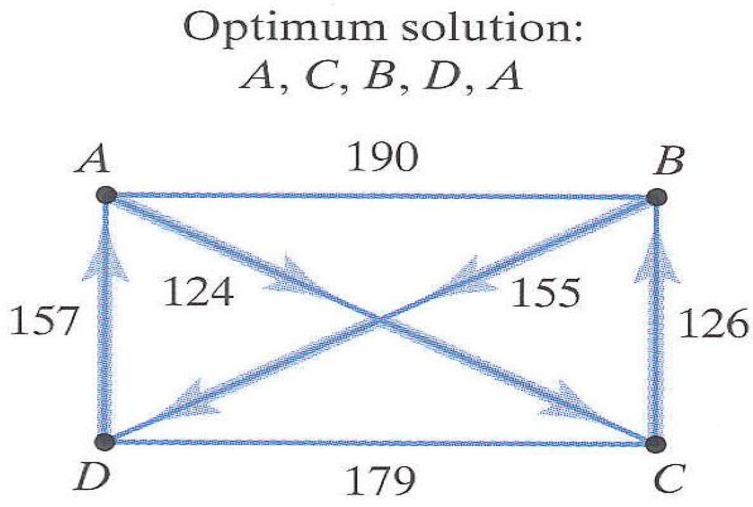
Minimum sums

Weighted Graphs and the Traveling Salesperson Problem

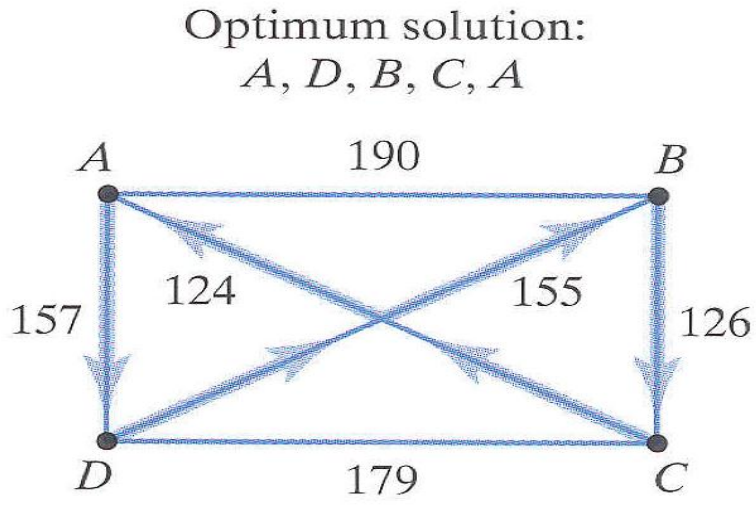
It is clear that there are two Hamilton circuits that have the minimum cost of $562. The optimal solution is either A,C,B,D,A or A,D,B,C,A.

For the sales director, this means that either route shown in Figures (a) and (b) has the least expensive way, to visit the regional offices in cities B, C, and D.

Notice that any of the two route solution is the reverse of the other.



(a) Fly from A to C to B to D and then back to A



(b) Fly from A to D to B to C and then back to A

Weighted Graphs and the Traveling Salesperson Problem

Suppose that a supercomputer can find the sum of the weights of one billion, or 10^9 Hamilton circuits per second. Because there are 31,536,000 seconds in a year, the computer can calculate the sums for approximately 3.2×10^{16} Hamilton circuits in one year. The table below shows that as the number of vertices increases the Brute Force Method is useless even with a powerful computer.

Number of Vertices	Number of Hamilton Circuits	Time Needed by a Supercomputer to Find Sums of All Hamilton Circuits
18	$17! \approx 3.6 \times 10^{14}$	$\approx$0.01 year $\approx$ 3.7 days
19	$18! \approx 6.4 \times 10^{15}$	$\approx$0.2 year $\approx$ 73 days
20	$19! \approx 1.2 \times 10^{17}$	$\approx$3.8 years
21	$20! \approx 2.4 \times 10^{18}$	$\approx$76 years
22	$21! \approx 5.1 \times 10^{19}$	$\approx$1597 years
23	$22! \approx 1.1 \times 10^{21}$	$\approx$35,125 years

Weighted Graphs And:

TRAVELING SALESMAN PROBLEM TSP

❖The Traveling Salesman Problem was first formulated a in 1930. It is a mathematical problem used in graph theory that requires one to find the most efficient route (tour) that a salesman can take to visit *n* cities exactly once and return home.

❖In general, the objective is to visit *n* cities once and return home with the minimum amount of travel. This relates to our project in that we must use the Traveling Salesman Problem in order to find the shortest possible route for a rover that will visit seven sites on Mars.

Weighted Graphs and the Traveling Salesperson Problem

Suppose a sales director who lives in city *A* is required to fly to regional offices in ten other cities and then return home to city *A*. With(11 - 1)!, or 3,628,800, possible Hamilton circuits, a list is out of the question.

What do you think of this option? Start at city *A*. From there, fly to the city to which the air fare is cheapest. Then from there fly to the next city to which the air fare is cheapest, and so on. From the last of the ten cities, fly home to city *A*. By continually taking an edge with the smallest weight, we can find approximate solutions to traveling s ales person problems. This method is called the Nearest Neighbor Method.

Weighted Graphs and the Traveling Salesperson Problem

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Weighted Graphs and the Traveling Salesperson Problem

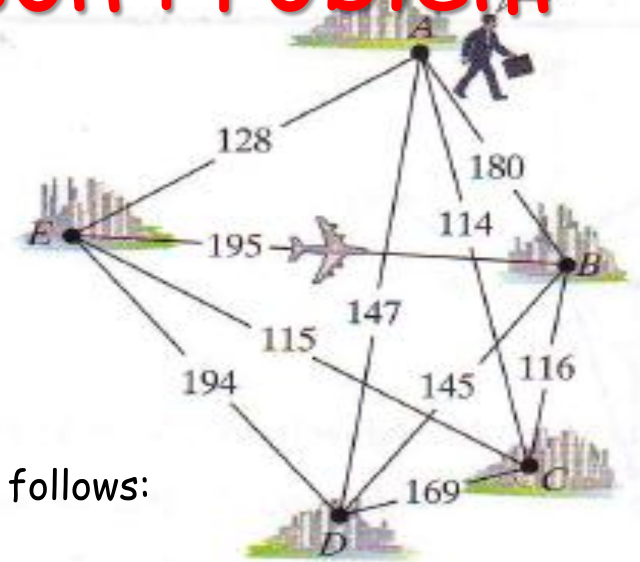
The Nearest Neighbor Method Of Finding Approximate Solutions to Travelling Salesperson Problems

The optimal solution can be approximated u sing the following steps:

1. Model the problem with a complete, weighted graph.
2. Identify the vertex that serves as the starting point.
3. From the starting point, choose the edge with the smallest weight. Move along this edge to the second vertex. (If there is more than one edge with the smallest weight, choose either one.)
4. From the second vertex, choose the edge with the smallest w eight that does not lead to a vertex already visited. Move along this edge to the third vertex.
5. Continue building the circuit, one vertex at a time, by moving along the edge with the smallest weight until all vertices are visited.
6. From the last vertex, return to the starting point.

Weighted Graphs and the Traveling Salesperson Problem

A sales director who lives in city A is required to fly to regional offices in cities B, C, D, and E. The weighted graph showing the one-way air fares is given in RHS.
Use the Nearest Neighbor Method to find an approximate solution. What is the total cost?



Solution: The Nearest Neighbor Method is carried out as follows:

1. Start at A.

2. Choose the edge with the smallest weight: 114. Move along this edge to C. (cost:$ 114)

3. From C choose the edge with the smallest weight that does not lead to A:115. Move along this edge to E. (cost: $115)

4. From ,E, choose the edge with the smallest weight that does not lead to a city already visited: 194. Move along this edge to D. (cost: $194).

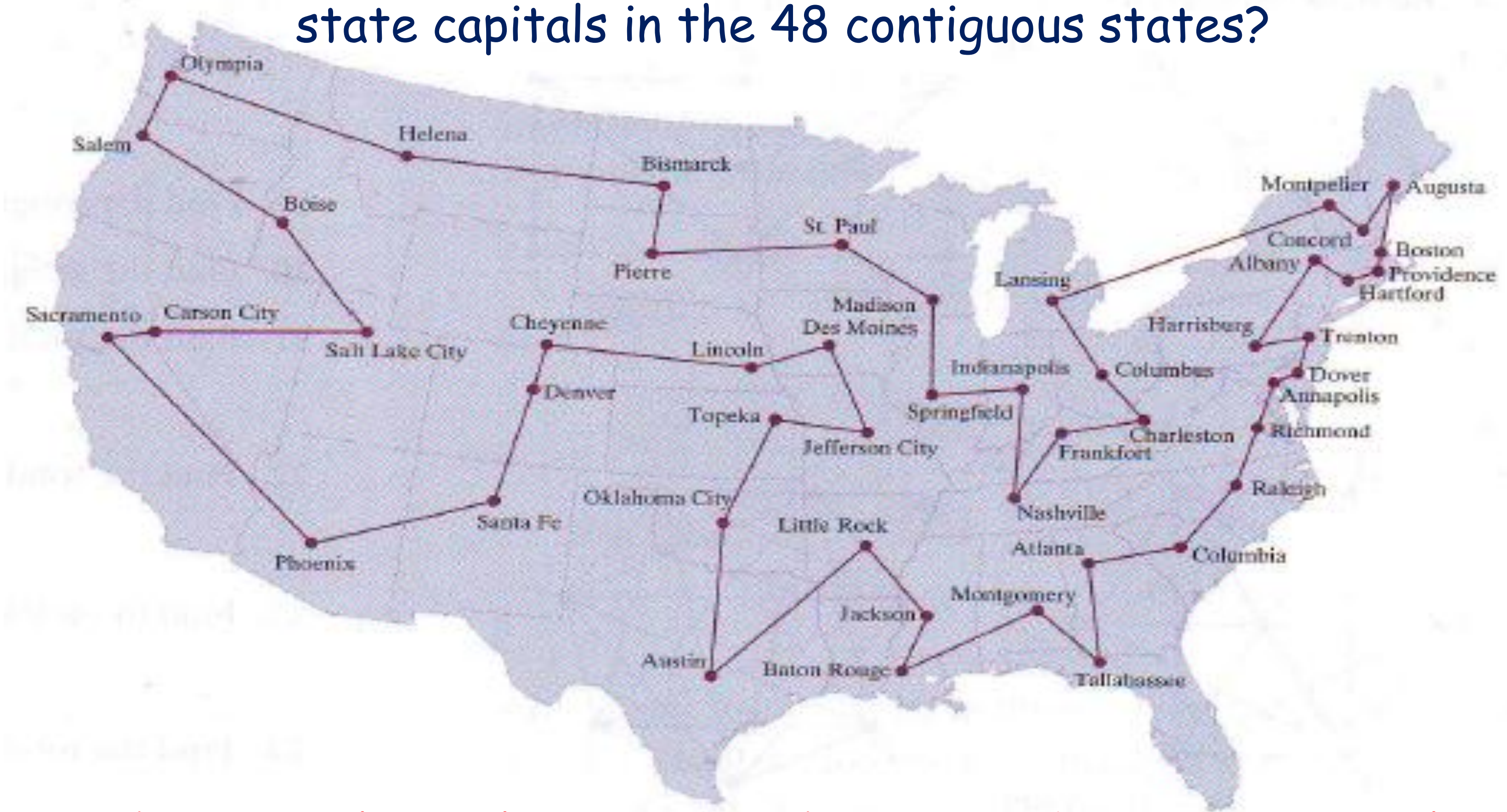
5. From D, there is little choice but to fly to B, the only city not yet visited. (cost: $ 145)

6. From B, close the circuit and return home to,4. (cost: $180)

An approximate solution is the Hamilton circuit: A,C,E,D,B,A.
The total cost is $114 + $115 + $194 + $145 + $180 = $748.

Weighted Graphs and the Traveling Salesperson Problem

What is the shortest Hamilton circuit that connects all of the state capitals in the 48 contiguous states?



The optimal Hamilton circuit, shown in red, is approximately 12,000 miles long.

EXPLORING MARS

4.D Hamiltonian Paths & Circuits

By:

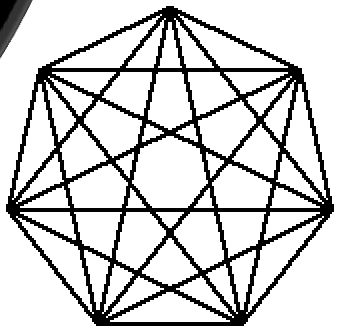
Christophe Dufour
Chrishon Adams
Mischael Joseph

MARS PROBE
مجس

❖Two Mars Exploration rovers/travellers were launched towards Mars on June 1o and July 7, 2003. Each landed on Mars on January 3 and January 24, 2004, respectively. The mission was to search for evidence of water on Mars. Also, their mission was to search for and characterize many rocks and soils that hold clues/signs to past/history water activity on Mars.

❖The spacecrafts were sent to targeted sites on opposite sides of Mars. Each site possibly holding evidence of having water in the past. The landing sites were Gusen Crater, a possible former lake, and Meridiani Planum, suggesting Mars had a wet past. The goal of each rover was to drive up to 40 meters (and 44 yards) per day, for a grand total of 1 kilometer (about three-quarters of a mile).

Weighted Graph On Mars



Distance Table

Point	A	G	H	I	N	P	W
A	______	7500	5000	2800	3500	1500	2200
G	7500	______	3000	6000	8000	6500	5000
H	5000	3000	______	4000	4800	3500	2800
I	2800	6000	4000	______	2000	3000	2900
N	3500	8000	4800	2000	______	4000	3200
P	1500	6500	3500	3000	4000	______	1300
W	2200	5000	2800	2900	3200	1300	______

Distance in miles

Hamilton Circuits

- We found the total number of different Hamilton circuits by using the following known property: *if N is the number of vertices of a complete graph then the number of Hamilton circuits in the graph is N minus one factorial:*

- N = 7 ⟹ (7-1)! = 6!
  - 6! = 6.5.4.3.2.1 = 720

- Therefore, there are 720 different Hamilton Circuits to find an optimal route for the seven sites on Mars.

BRUTE FORCE ALGORITHM

- Brute Force is a list of all the possible Hamilton circuits (tours) of the graph. For each Hamilton tour, we calculate its total weight (add the weights of all the edges in the circuit). An optimal tour (least value) is then chosen, there is always more than one optimal tour.

- The reason the Brute Force Algorithm is inefficient is that there are (6! = 6.5.4.3.2.1 = 720) different Hamilton Circuits to solve for an optimal tour.

Three Efficient Algorithms For Finding The Optimal Route

- The Cheapest Link Algorithm
  - CL = A → P → W → H → G → N → I → A = 21,400 miles
    - 1500 1300 2800 3000 8000 2000 2800
- The Nearest Neighbor Algorithm
  - N: A → P → W → H → G → I → N → A = 20,100 miles

    1500 1300 2800 3000 6000 2000 3500

- **The Nearest Neighbor Algorithm gives the most efficient route for the MARS probe to travel with a total of 20,100 miles.**

Repetitive Nearest Neighbor

Using the Repetitive Nearest Neighbor Algorithm, we found that *beginning* at site A produces the shortest traveling distance.

A → P → W → H → G → I → N → A = 20,100

P → W → A → I → N → H → G → P = 23,600

W → P → A → I → H → N → G → W = 27,400

N → I → A → P → W → H → G → N = 21,400

I → N → W → P → A → H → G → I = 27,000

H → W → P → A → I → N → G → H = 21,400

G → H → W → P → A → I → N → G = 21,400

Graph Coloring

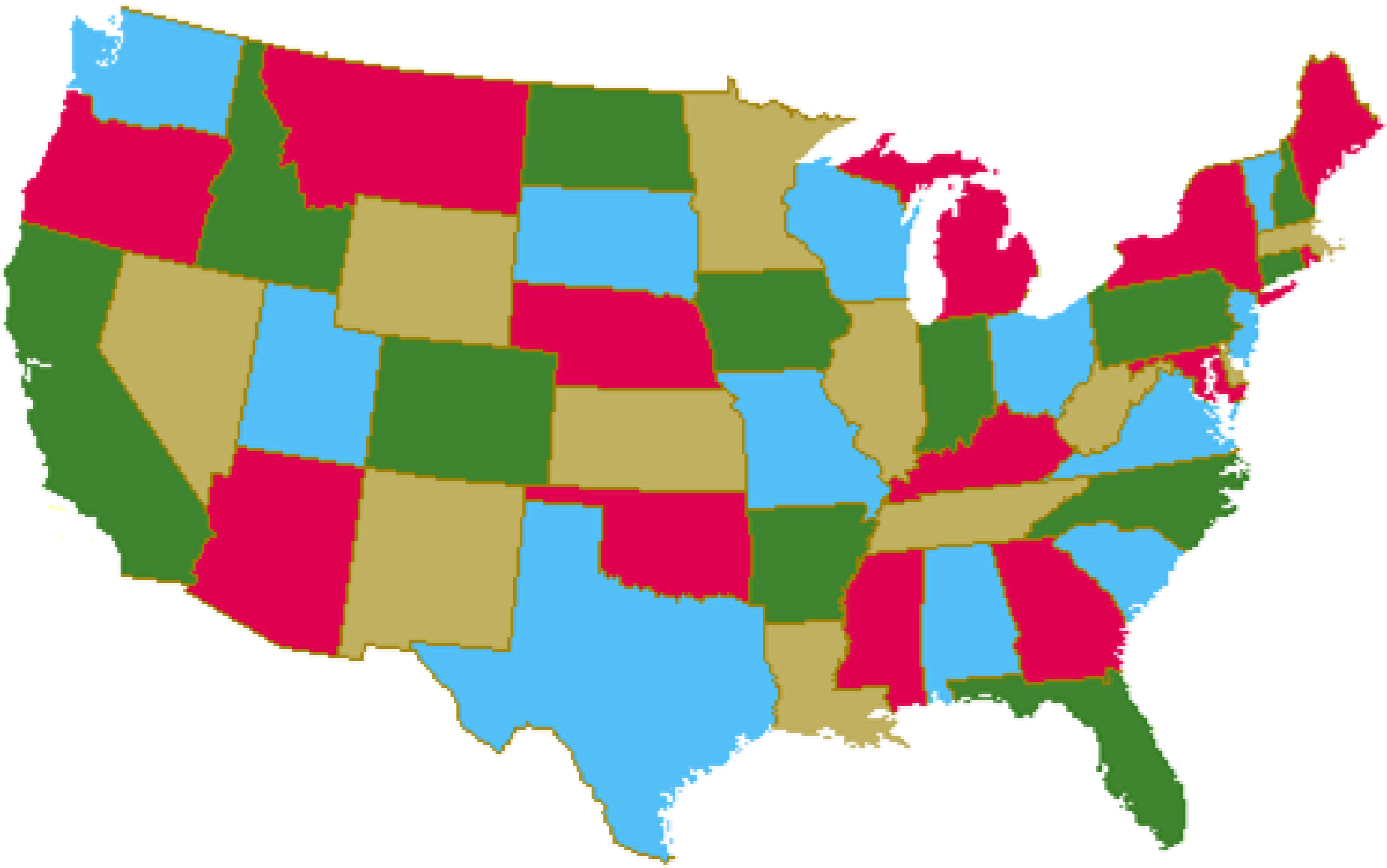
- Find the chromatic number of a graph
- Apply graph colorings in solving various problems

The Four Colour Theorem/Problem

Proposed by Francis Guthrie in 1852 and remained unsolved for more than a century.

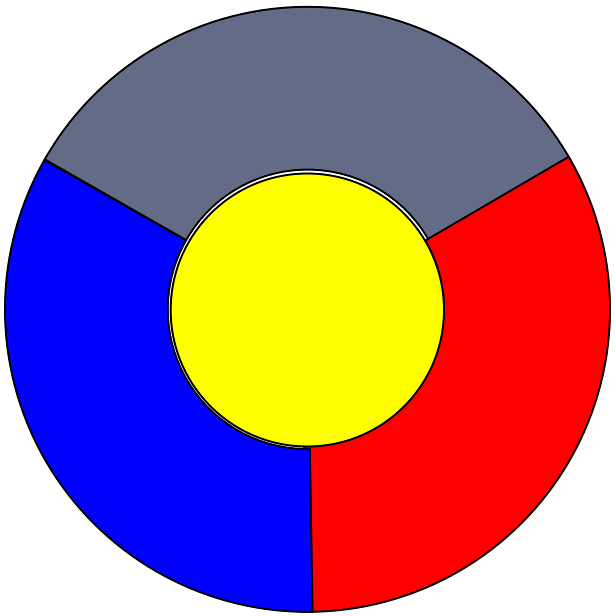
Can any map be coloured with 4 colours so that no two adjacent regions have the same colour?

Examples of a-4 Coloring Theorem



Why not 3 colours?

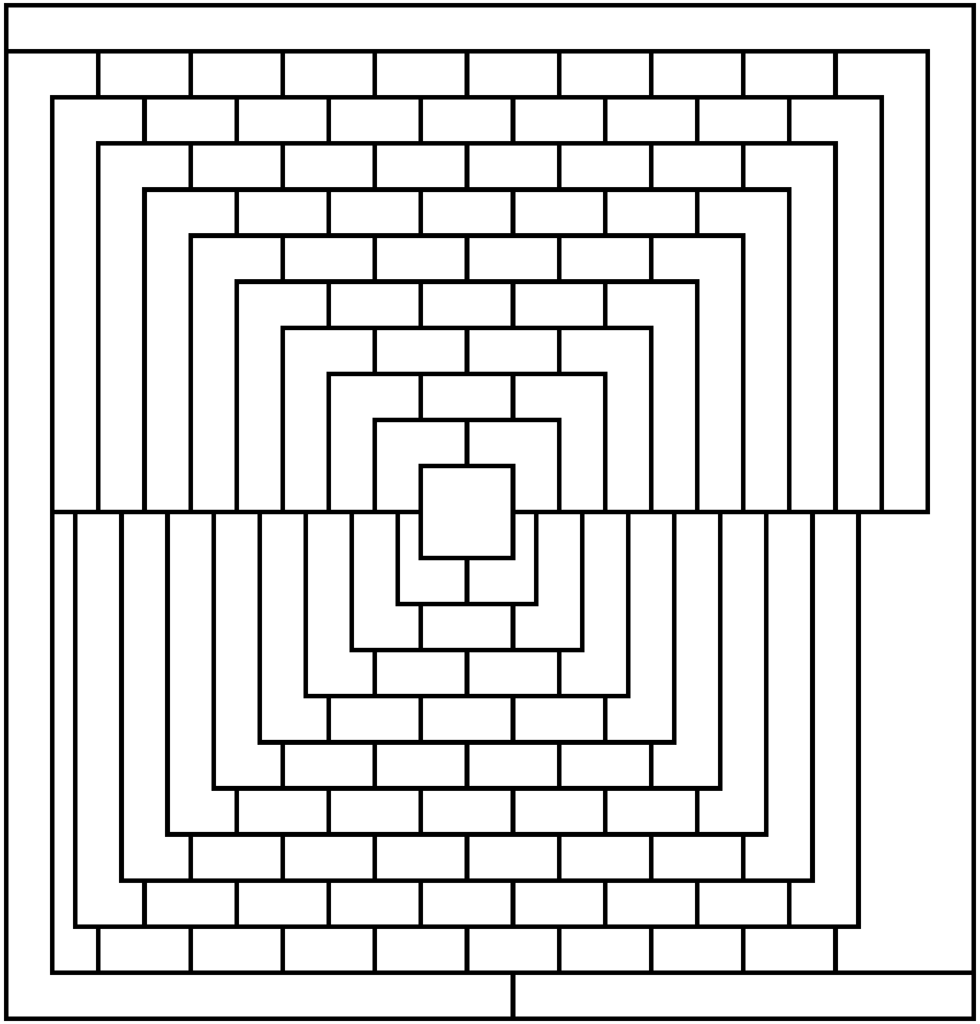
- A simple example shows that it impossible to always colour a map with only 3 colours.



Why not 5 colours?

- It was proved by 1890 that every map can be coloured with at most 5 colours.
- The difficult part of the problem was to show that there was no map sufficiently complicated as to need 5 colours.
- Martin Gardner set the following graph as a problem to his readers. Can you colour it using only 4 colours?

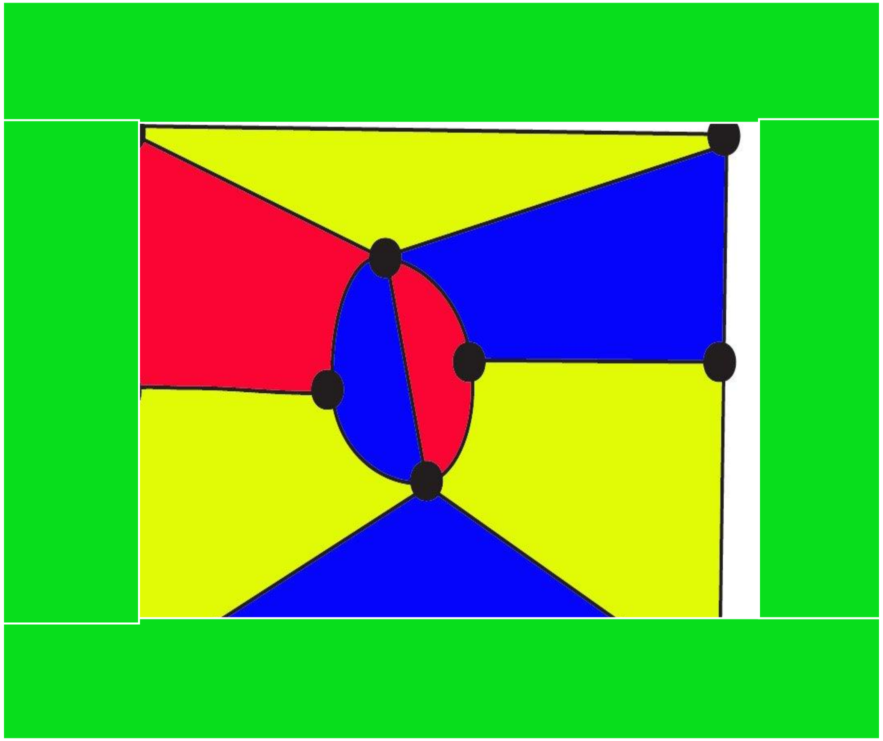
Martin Gardner's map

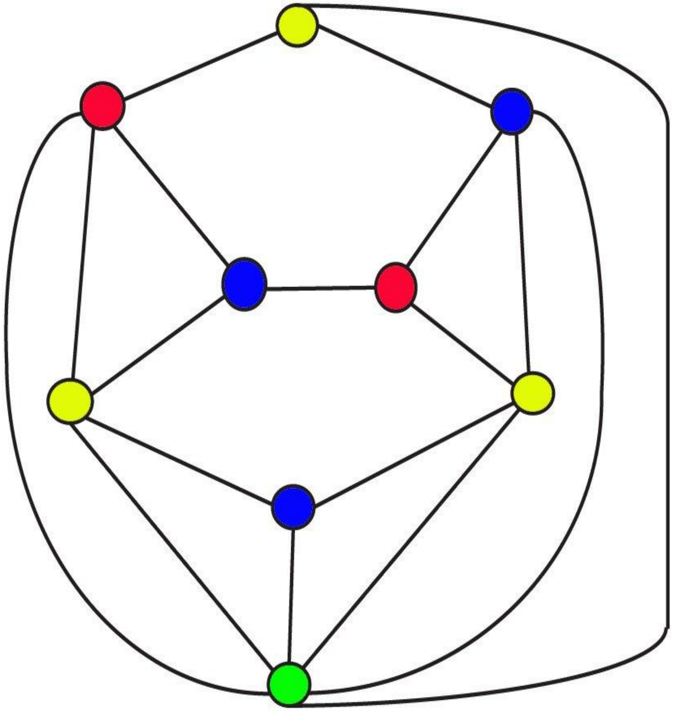


In Terms of Graphs

- The 4-colour problem can be phrased in terms of graphs.
- Each region of the map becomes a **node**, with two nodes being connected by an edge if and only if the regions are adjacent on the map.
- The problem becomes: can you colour the nodes with 4 colours so that an edge never connects two nodes of the same colour?

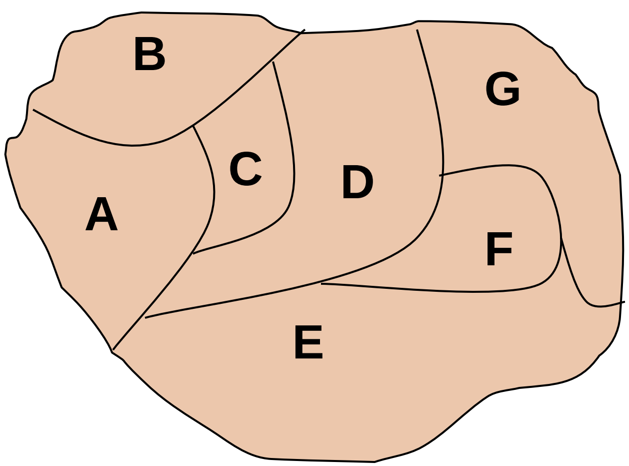
Maps to graphs (Dual Graph)



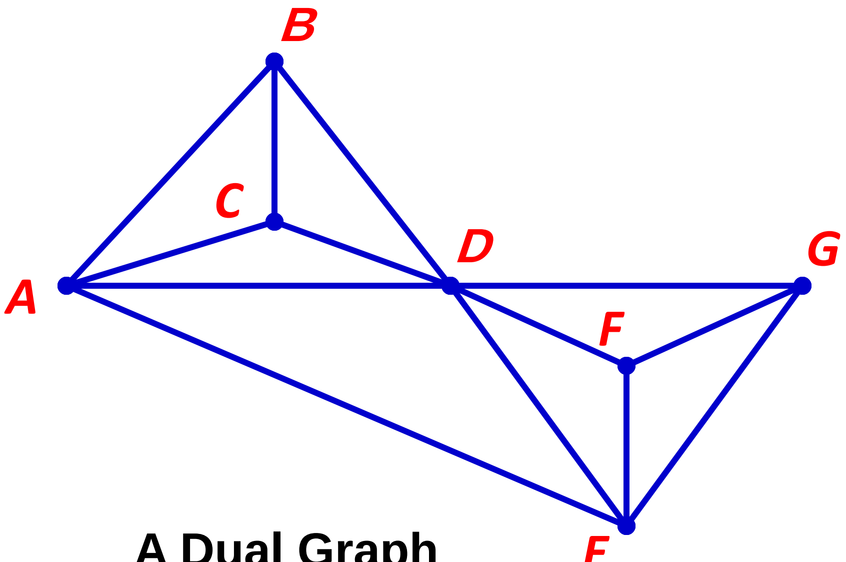


Maps and Dual Graphs

- Each map in the plane can be represented by a graph (Dual Graph).
  - Vertex: Region
  - Edge: A common border of two adjacent Regions A and B
- Dual Graph is planar graph



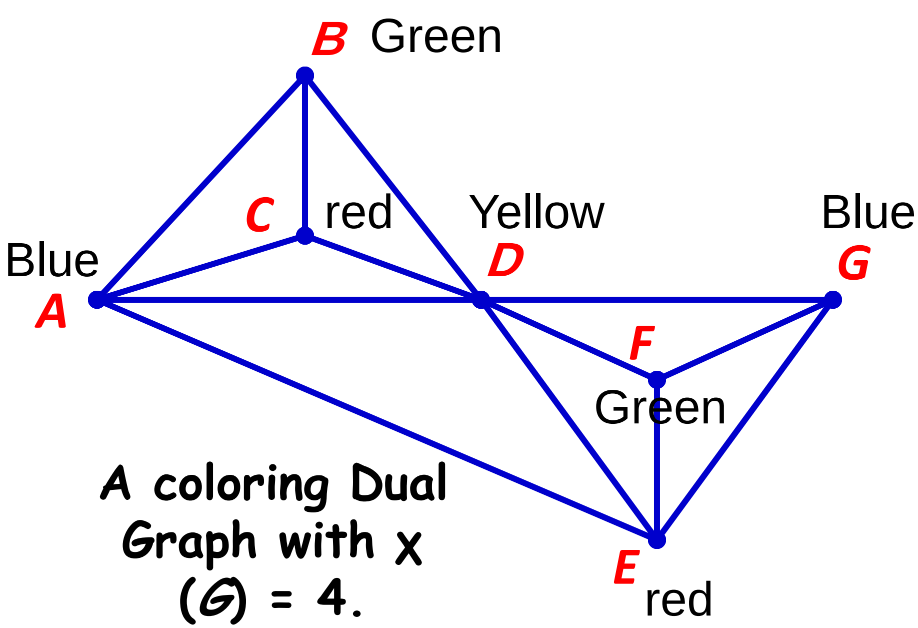
A map



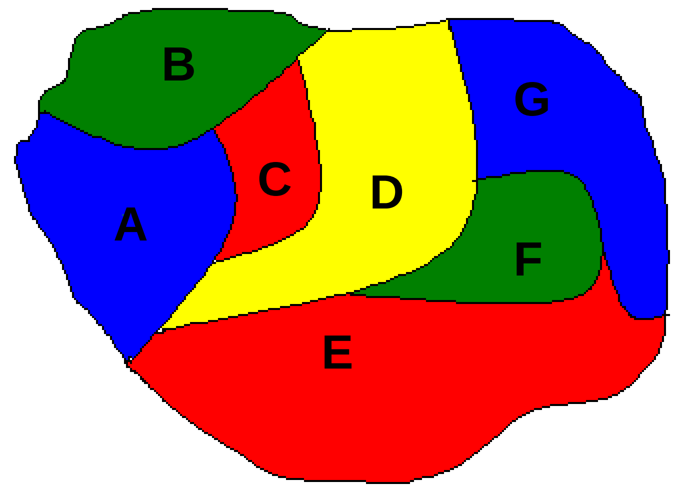
A Dual Graph

Graph Coloring

- The problem of coloring the regions of a maps is equivalent to the problem of coloring the vertices of the dual graph so that no two adjacent vertices in the graph have the same color.
- The least number of colors needed for a coloring graph is given by chromatic number which denoted by $\chi(G)$.
- The Chromatic number for planar graph is not greater than 4.



A coloring Dual Graph with $\chi(G) = 4$.



A coloring map

Check Point

Construct the dual graph for the map shown. Then find the number of colors needed to color the map so that no two adjacent regions have the same color.

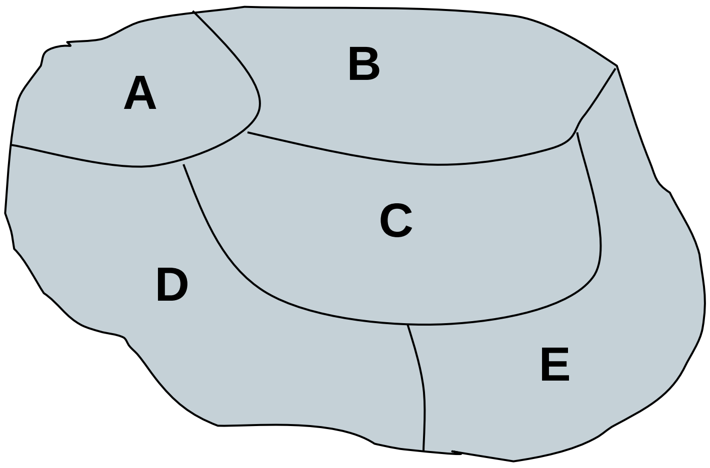


Figure 1

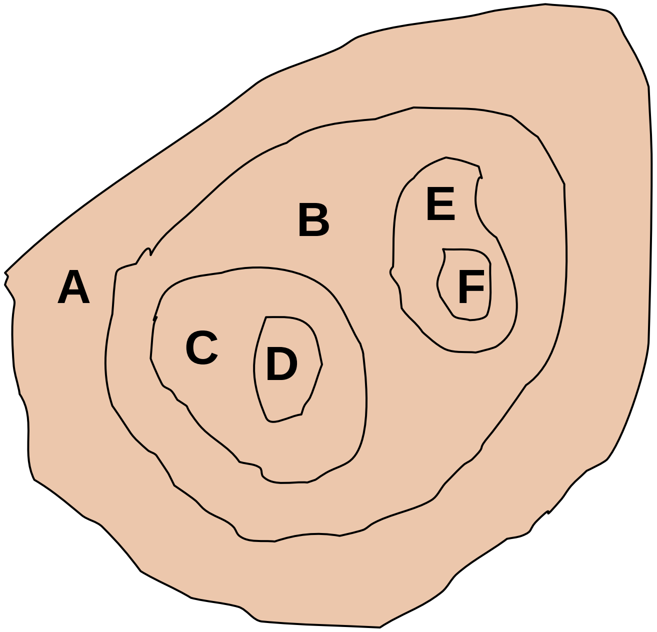


Figure 2

Check Point

Find the chromatic number of the given graphs.

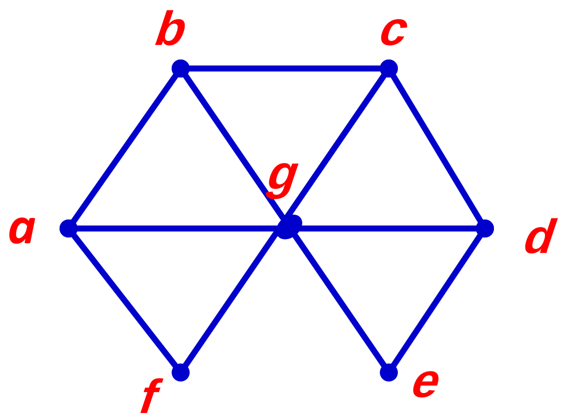


Figure 1

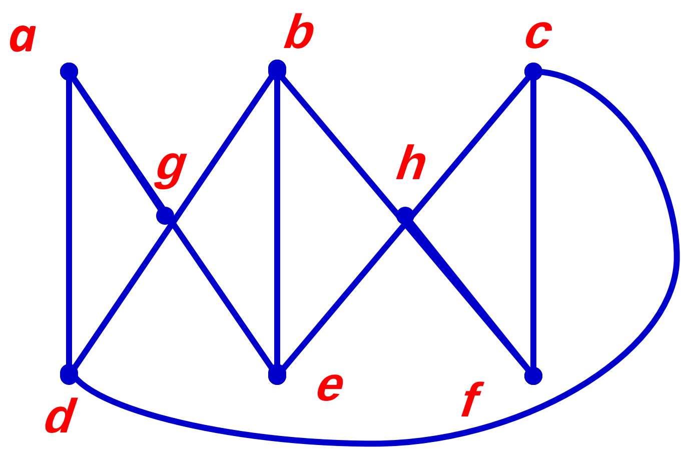
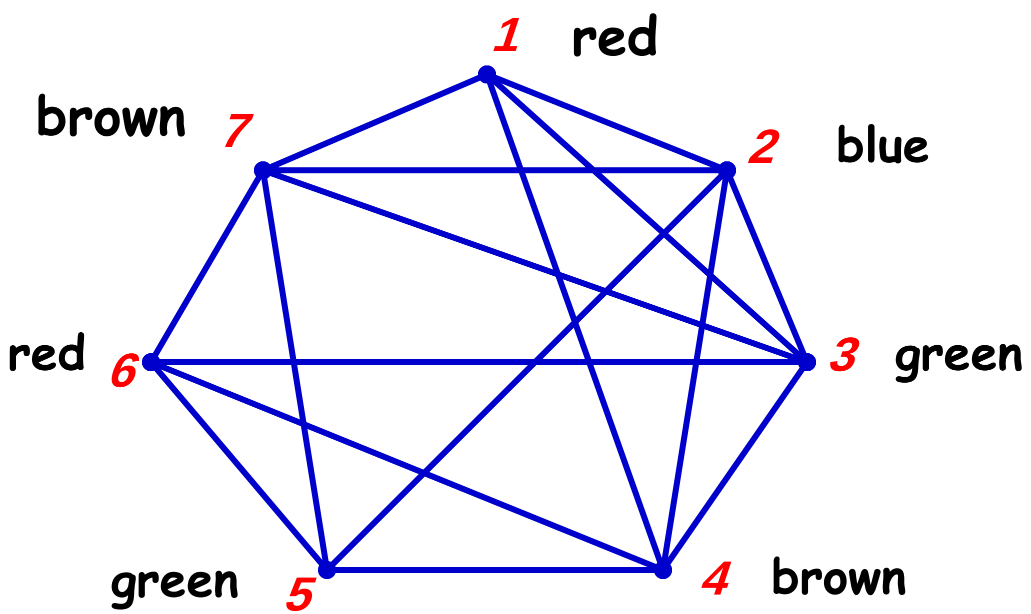


Figure 2

Scheduling Final Exams

How can the final exams at UMP be scheduled so that no student has two exams at the same time?

Suppose that, 7 finals to be scheduled and the following pairs of course have common students.
1&2, 1&3, 1&4, 1&7, 2&3, 2&4, 2&5, 2&7, 3&4, 3&6, 3&7, 4&5, 4&6, 5&6, 5&7, and 6&7



Time Period	Course
I	1, 6
II	2
III	3, 5
IV	4, 7

Check Point

Schedule the final exams for Math 115, Math 116, Math 185, Math 195, CS 101, CS 102, CS 273, and CS 473, using the fewest number of different slots, if there are no students taking both Math 115 and CS 473, Math 116 and CS 473, both Math 195 and CS101, both Math 195 and CS 102, both Math 115 and Math 116, both Math 115 and Math 185, and both Math 185 and Math 195, but there are students in every other combination of courses.

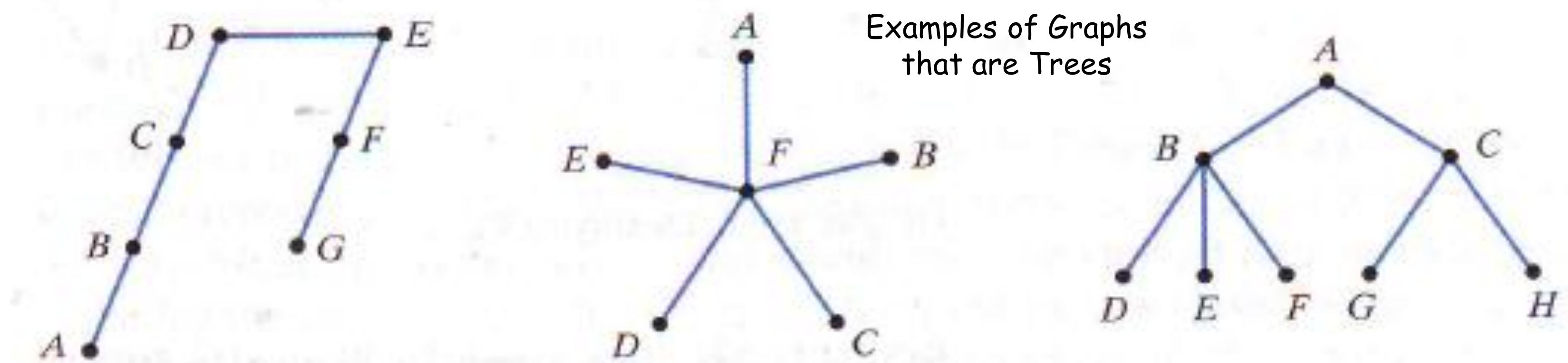
5. Trees

- Understand the definition and Properties of a tree.
- Extraction of spanning tree from a given graph.
- Applications of trees in encoding and decoding

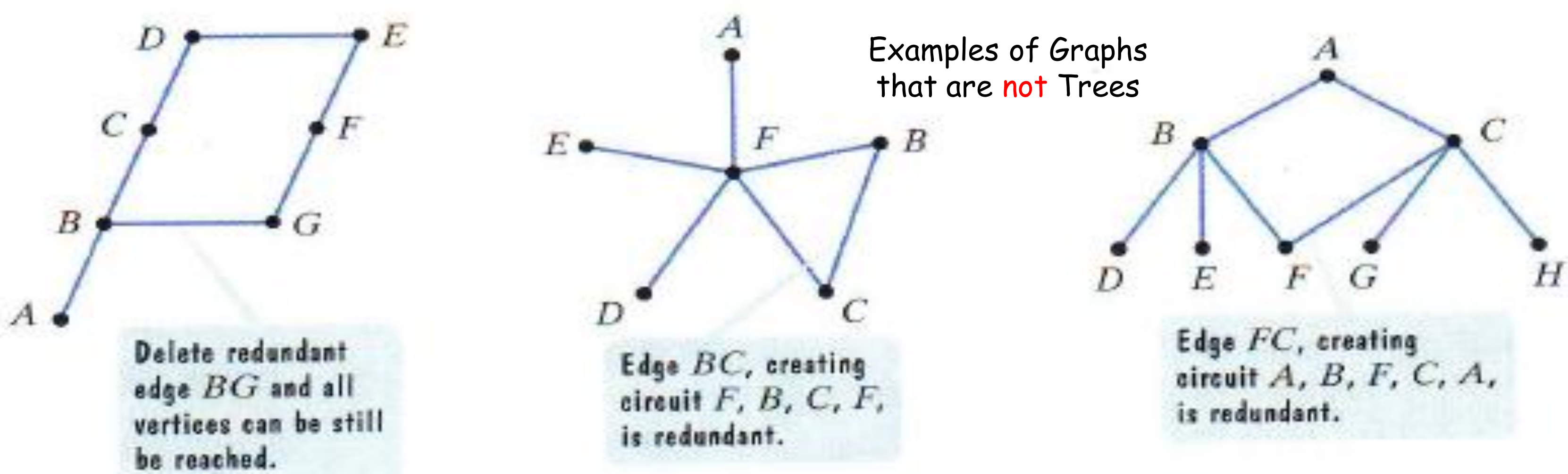
5.A Definitions & Properties

- A tree is a graph that is connected and has no circuits. All trees have the following properties:

1. There is one and only one path joining any two vertices.
2. Every edge is a bridge.
3. A tree with n vertices must have n - 1 edges.



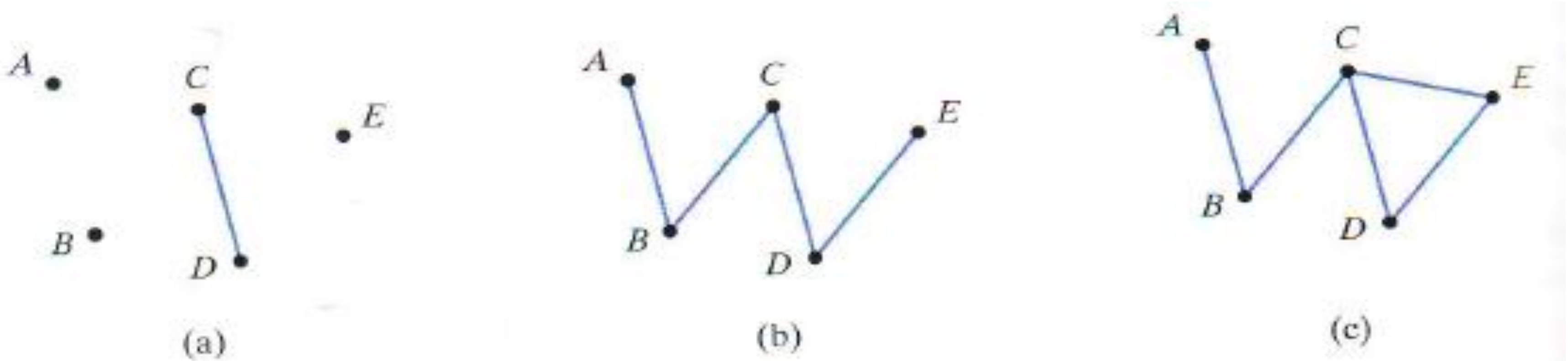
Examples of Graphs that are Trees



Examples of Graphs that are not Trees

Illustrative Example

which of the following graphs is a tree? Explain why the other two graphs shown are not trees?



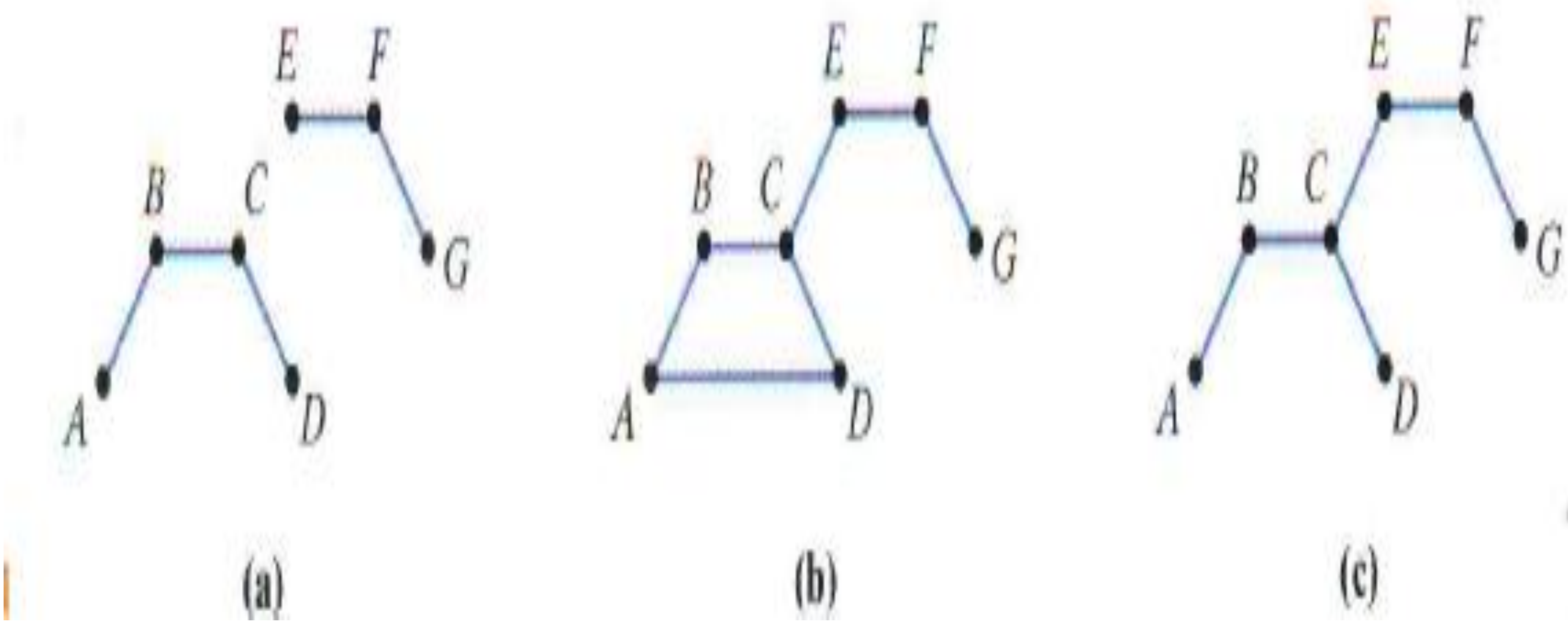
Solution The graph in Figure (b) is a tree. It is connected and has no circuits. There is only one path joining any two vertices. Every edge is a bridge; if removed, each edge would create a disconnected graph. Finally, the graph has 5 vertices and 5 - 1, or 4, edges.

The graph in Figure (a) is not a tree because it is disconnected. There are five vertices and only one edge; a tree with five vertices must have four edges.

The graph in Figure (c) is not a tree because it has a circuit, namely C, D, E, C. There are five vertices and five edges; a tree with five vertices must have four edges.

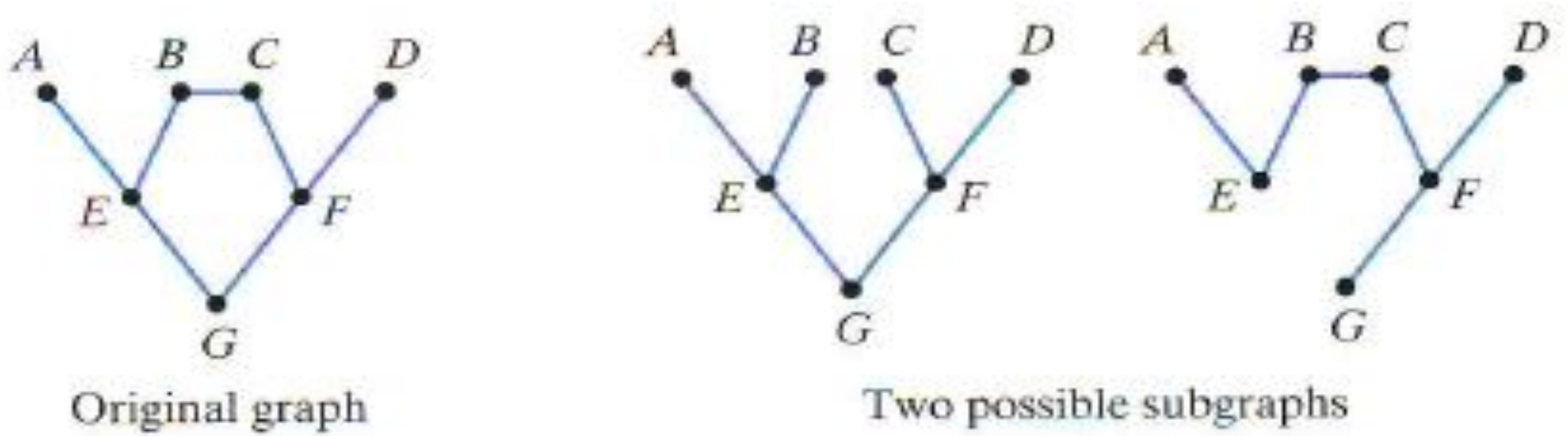
Check Point

Which graph in the following Figure is a tree? Explain why the other two graphs shown are not trees.



5.A Definitions & Properties of Trees

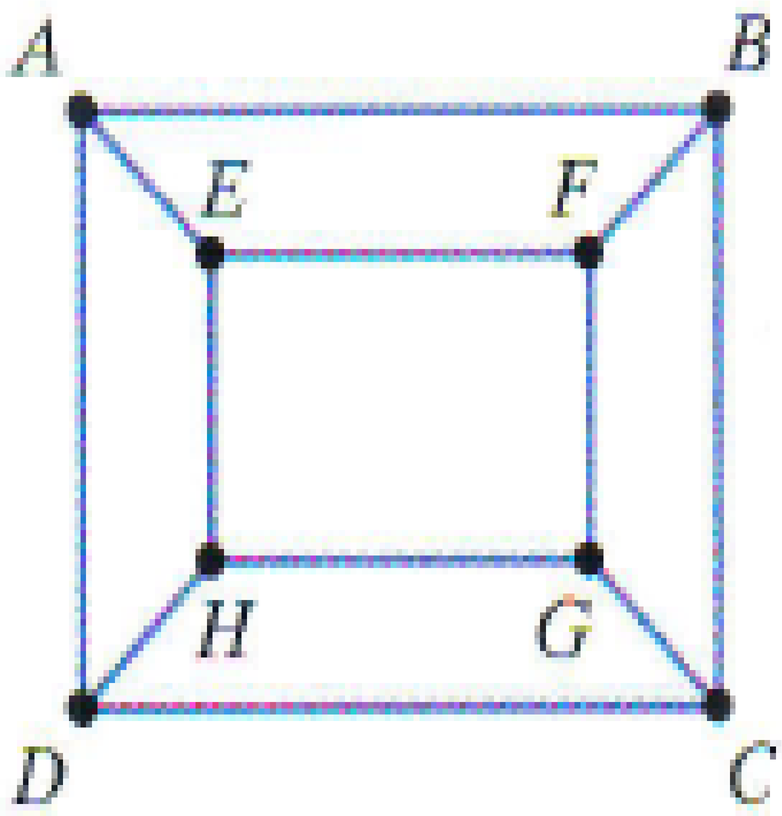
5.B Spanning Tree

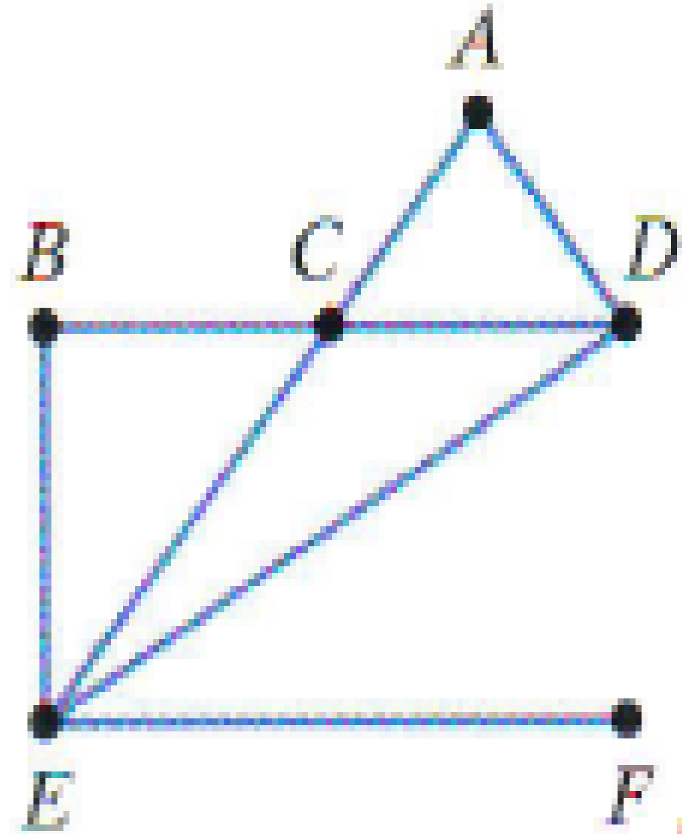


- A subgraph that contains all of a connected graph's vertices is connected and contains no circuits is called a spanning tree. The two subgraphs in Figure given above, are spanning trees for the original graph. By removing redundant connections the spanning trees increase the efficiency of the network modeled by the original graph.
- It is possible to start with a connected graph, retain all of its vertices, and remove edges until a spanning tree remains. Being a tree, the spanning tree must have one less edge than it has vertices.
- Spanning Tree is one way to increase the efficiency of a network byremoving redundant connection

Check Point

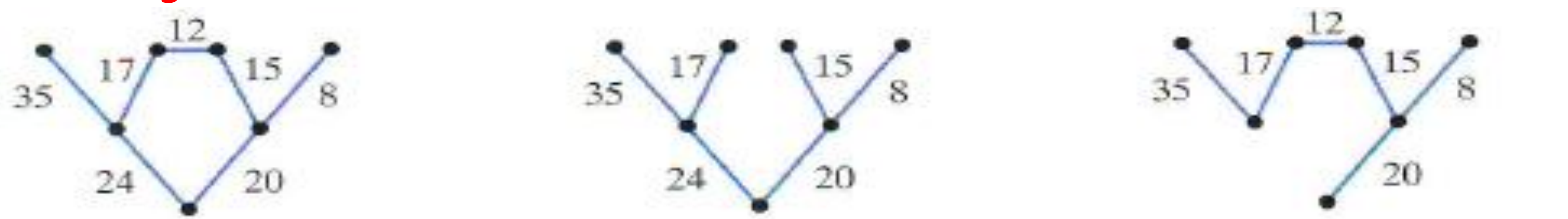
Find a spanning tree for each of the graphs given below?





Minimum Spanning Tree

Many applied problems involve creating the most efficient network for a weighted graph. The weights often model distance costs or time, which we want to minimize. We do this by finding a minimum spanning tree. The minimum spanning tree for a weighted graph is a spanning tree with the smallest possible total weight.



(a) Original weighted graph

(b) A spanning tree with weight $35 + 24 + 20 + 8 + 17 + 15 = 119$

(c) A spanning tree with weight $35 + 17 + 12 + 15 + 20 + 8 = 107$

Figures (b) and (c) show two spanning trees for the weighted graph in Figure (a). The total weight for the spanning tree in Figure (c), 107, is less than that in Figure (b), 119. Is this the minimum spanning tree, or should we continue to explore other possible spanning trees whose total weight might be less than 107?

A very simple graph can have many spanning trees. Finding the minimum spanning tree by finding all possible spanning trees and comparing their weights would be too time-consuming

Minimum Spanning Tree Using Kruskal's Algorithm

In 1956, the American mathematician Joseph Kruskal discovered a procedure that will always yield a minimum spanning tree for a weighted graph. The basic idea in Kruskal's Algorithm is to always pick the smallest available edge but avoid creating any circuits.

Kruskal's Algorithm

Here is a procedure for finding the minimum spanning tree from a weighted graph:

1. Find the edge with the smallest weight in the graph. If there is more than one, pick one at random. Mark it in red (or using any other designation).

2. Find the next-smallest edge in the graph. If there is more than one, pick one at random. Mark it in red.

3. Find the next-smallest unmarked edge in the graph that does not create a red circuit. If there is more than one, pick one at random. Mark it in red.

4. Repeat step 3 until all vertices have been included. The red edges are the desired minimum spanning tree

Illustrative Example

Fig. a

Fig. b

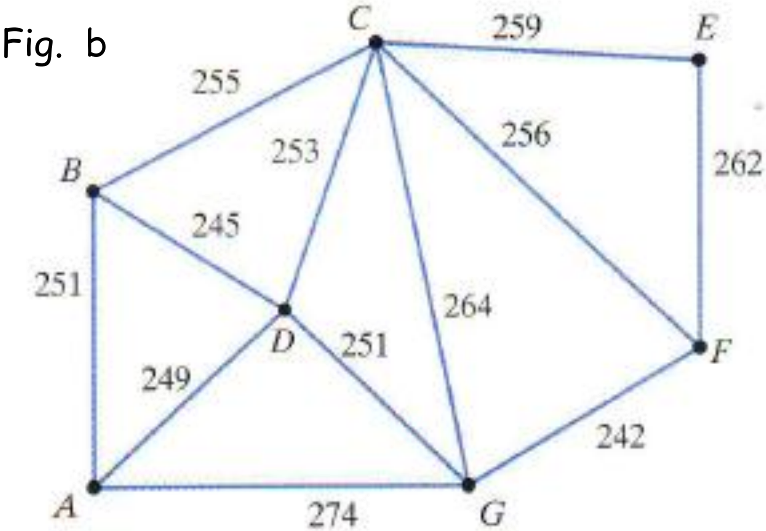


Figure (a) shows seven buildings on a college campus are connected by the sidewalks .

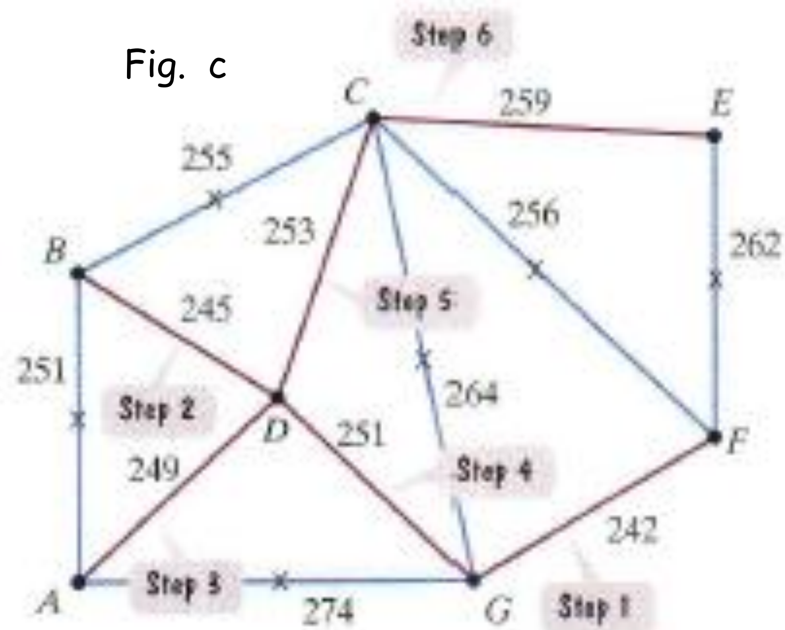
The weighted graph in Figure (b) represents buildings as vertices, sidewalks as edges, and sidewalk lengths as weights. A heavy snow has fallen and the

sidewalks need to be cleared quickly. Campus services decides to clear as little as possible and still ensure that students walking from building to building, will be able to do so along cleared paths. Determine the shortest series of sidewalks to clear. What is the total length of the sidewalks that need to be c1eared?

Illustrative Example

Solution Campus services wants to keep the total length of cleared sidewalks to a minimum and still have a cleared path connecting any two buildings. Thus they are seeking a minimum spanning tree for the weighted graph in Figure (b). We find this minimum spanning tree using Kruskal's Algorithm. Refer to Figure (c) as one reads the steps in the algorithm.

Fig. c



Step 1. Find the edge with the smallest weight. Select edge GF (length: 242 feet) by marking it in red.

Step 2. Find the next-smallest edge in the graph. Select edge BD (length: 245 feet) by marking it in red.

Step3. Find the next-smallest edge in the graph. Select edge AD (length: 249feet) by marking it in red.

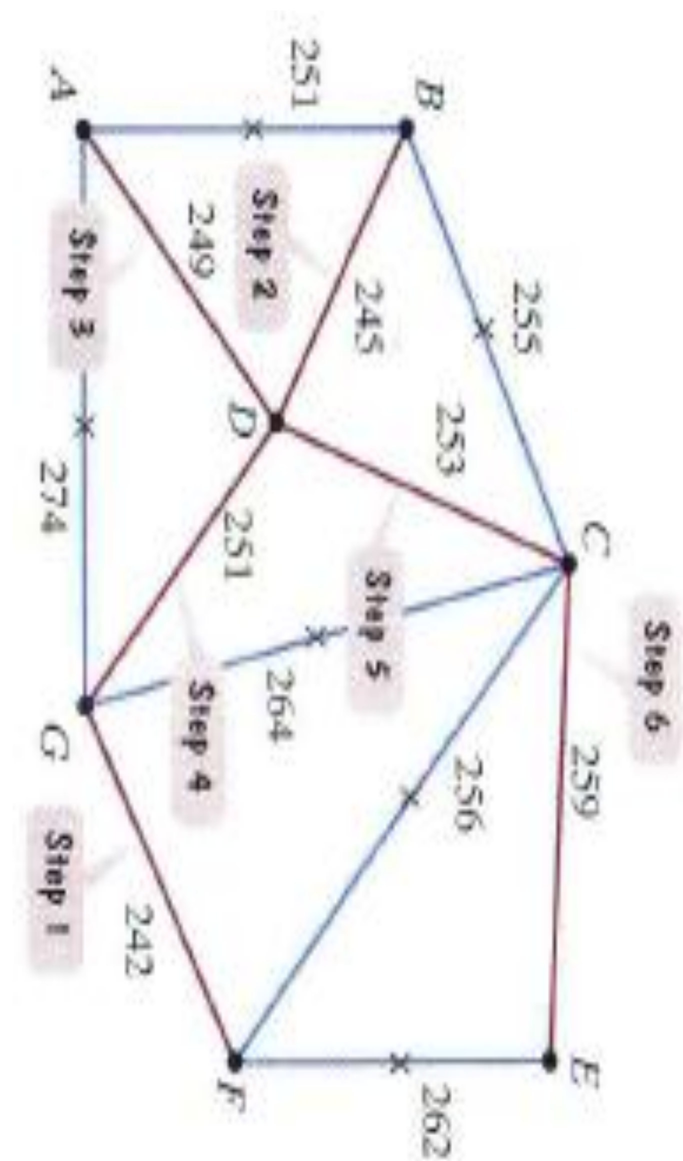
Step 4. Find the next-smallest edge in the graph that does not create a circuit. The next-smallest edges are AB and DG (length of each: 25I feet). Do not select AB-it creates a circuit. Select edge DG by marking it in red.

Illustrative Example

Step 5. Find the next-smallest edge in the graph that does not create a circuit. Select edge CD (length: 253 feet) by marking it in red. Notice that this does not create a circuit'

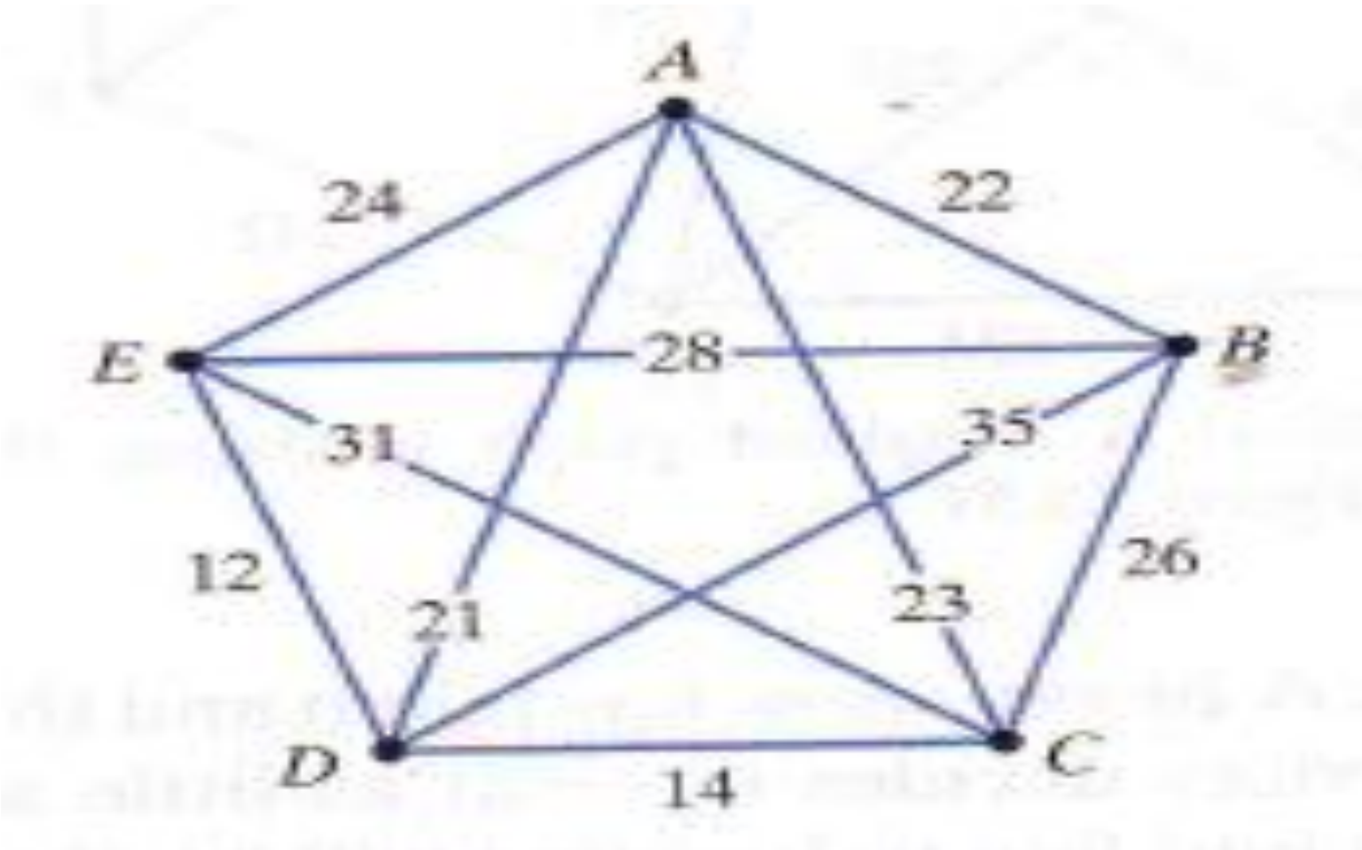
Step 6. Find the next-smallest edge in the graph that does not create a circuit. The next-smallest edge is BC, (length 255 feet), but this creates a circuit. Discard BC. The next-smallest edge is CF, (length 256 feet), but this creates a circuit. Discard CF. The next.-smallest edge is CE (length: 259 feet). This does not create a circuit, so select edge CE by marking it in red.

Now one can see the minimum spanning tree in Figure (d) is completed? The red subgraph contains all of the graph's seven vertices' is connected, contains no circuits, and has 7 - 1, or 6, edges. Therefore, the red subgraph in Figure (d) shows the shortest series of sidewalks to clear. From the figure one see that there are: 242 + 245 + 249 + 251 + 253 + 259 or 1,499 feet of sidewalks that need to be cleared



Check Point

Use Kruskals' Algorithm to find the minimum spanning tree for the graph shown below. Give the total weight of the minimum spanning tree.

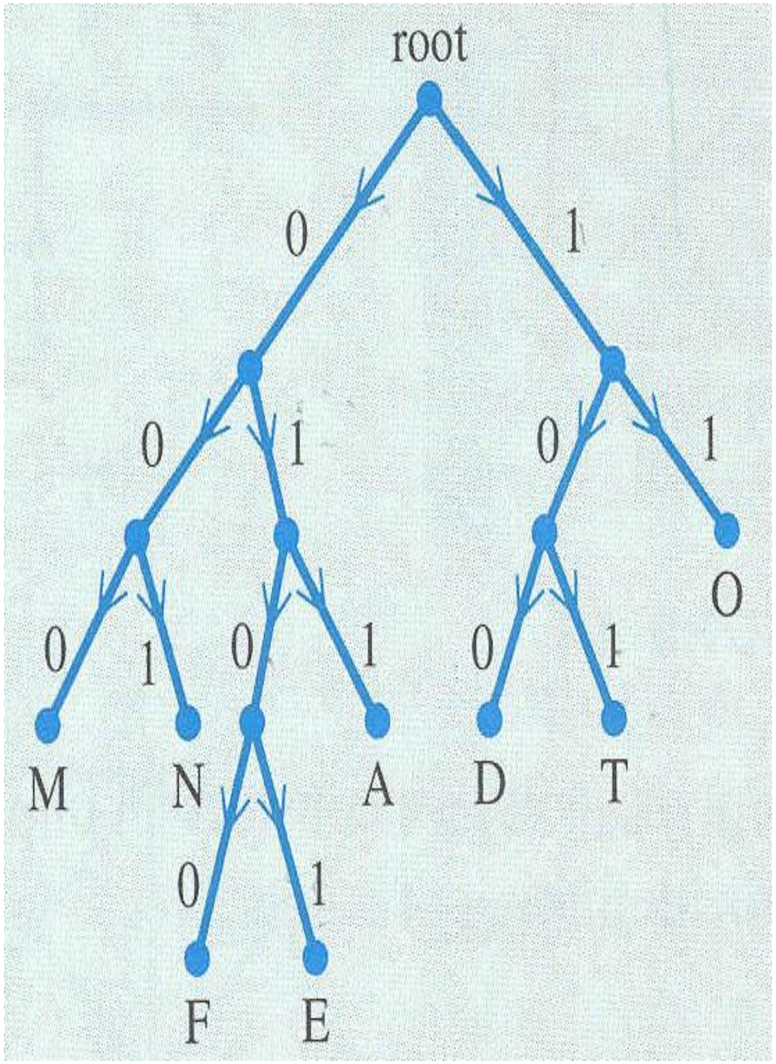


5.C Binary Encoding Tree

The unique path property of a tree can be used to set up a code. Here we set up a binary code, that is, a code with strings of 0s and 1s representing letters. (Recall that when we represent numbers in binary form we use only the symbols 0 and 1.)

To set up the binary code we use a special kind of tree (called a binary tree) like that shown in the figure. This tree is a directed graph (there are arrows on the edges). The vertex at the top (with no arrows pointing toward it) is called the root of the directed tree; the vertices with no arrows pointing away from them are called leaves of the directed tree.

We label each leaf with a letter we want to encode.

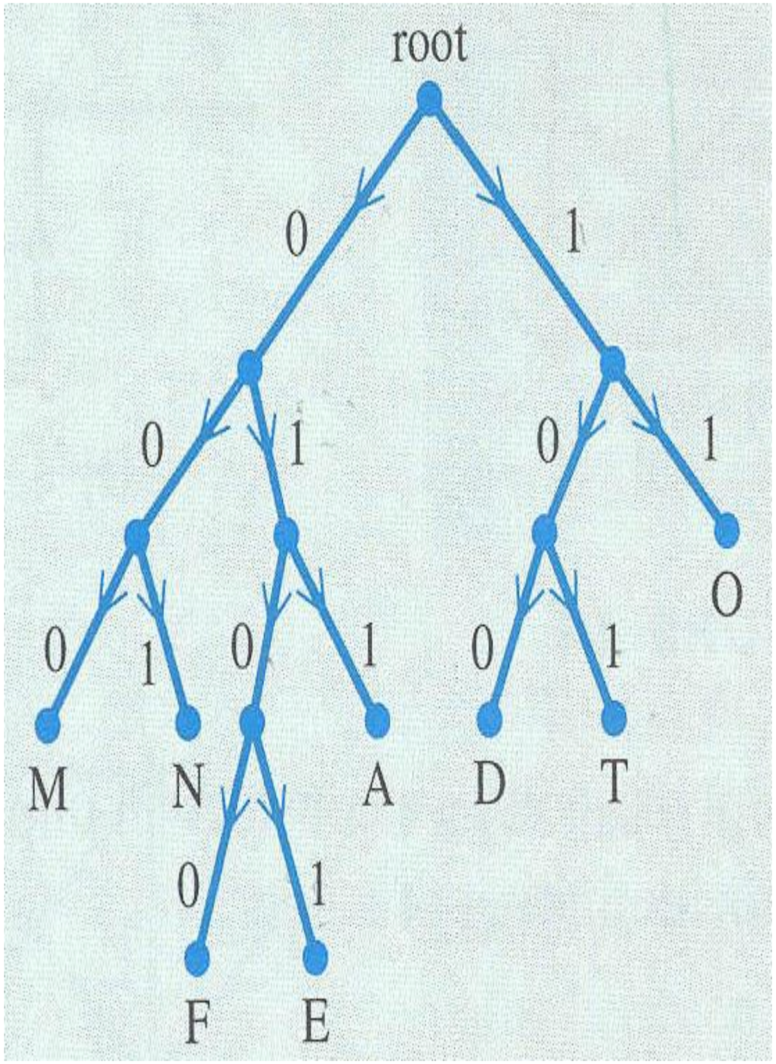


5.C Binary Encoding Tree

The diagram shown provides an encoding for only 8 letters, but we could easily draw a bigger binary tree with more leaves to represent more letters.

We now write a 0 on each branch extending to the left and a 1 on each branch extending to the right.

To show how the encoding works, let us write the word MAD using the code. Follow the unique path from the root of the tree down to the appropriate leaf, noting in order the labels on the edges.



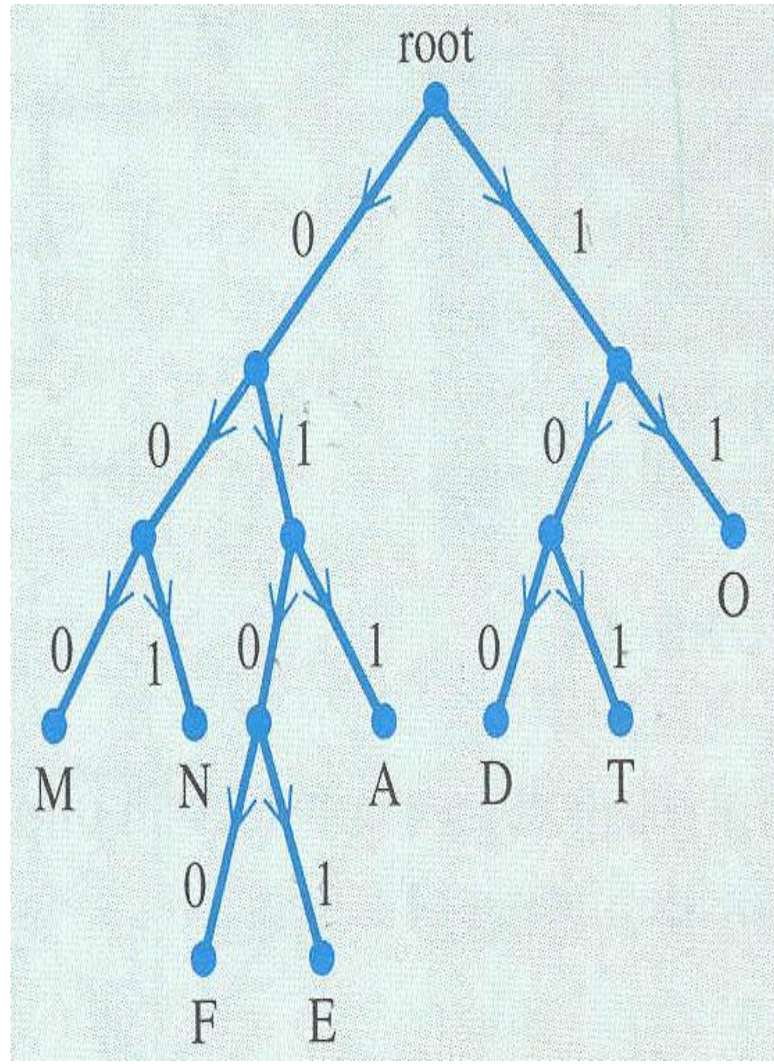
5.C Binary Encoding Tree

M is written 000.

A is written 011.

D is written 100.

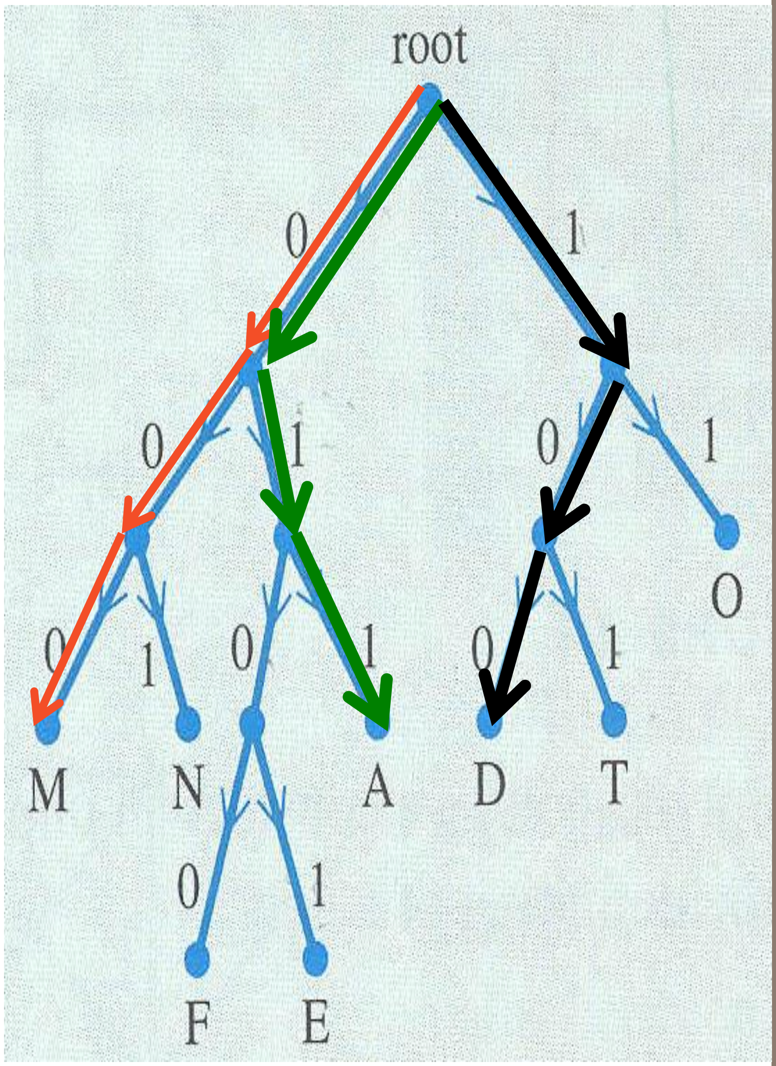
So MAD is written 000011100.



5.C Binary Encoding Tree

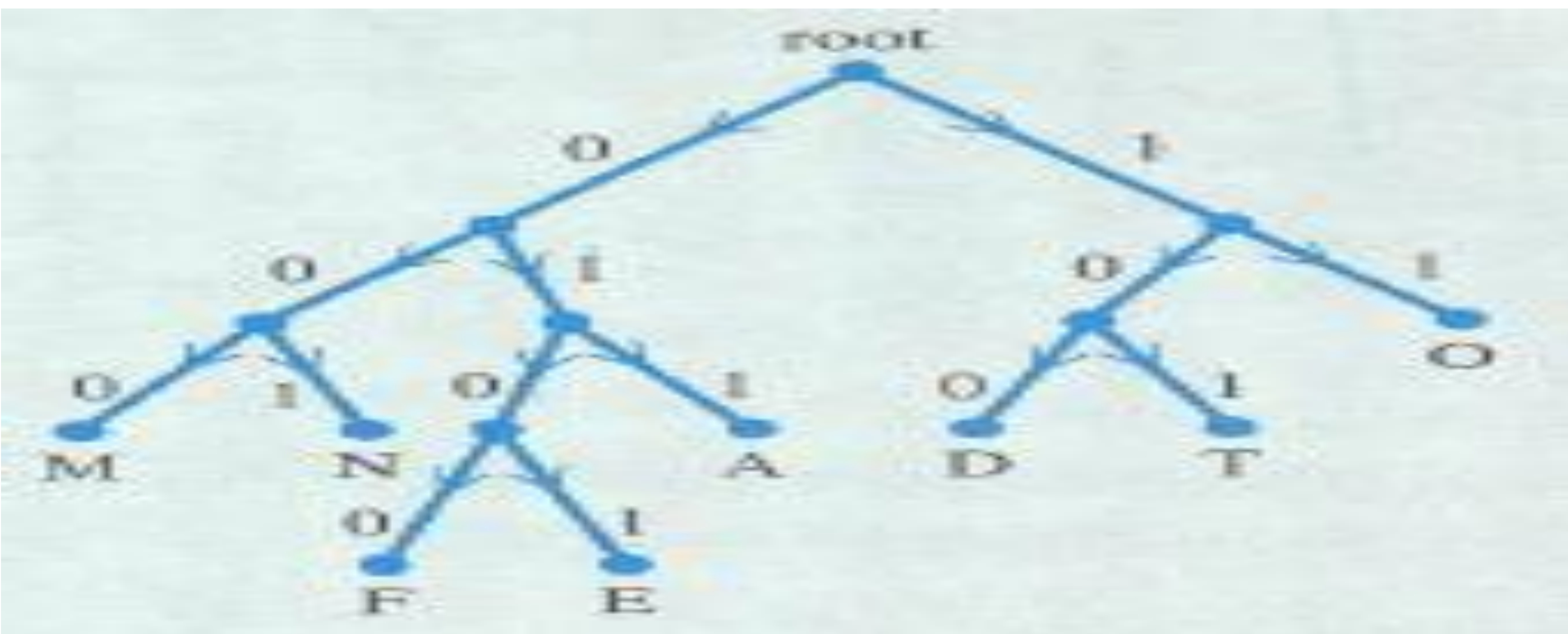
We can easily translate this code using our tree. Let us see how we could decode 000011100. Referring to the tree, we can see that there is only one path from the root to a leaf that can give rise to those first three 0s, and that is the path leading to **M**. So we can begin to separate the code word into letters: 000-011100. Again following down from the root, the path 011 leads us unambiguously to **A**. So we have 000-011-100. The path 100 leads unambiguously to **D**.

The reason we can translate the string of 0s and 1s back to letters without ambiguity is that no letter has a code that is the start of the code for a different letter.



Check Point

1. Use the binary encoding tree to write the binary code for each of the following words: FEET, ANT
2. Use the encoding tree to find the word represented by each of the following codes: 1000101001101 , 0001111001
3. Decode the following message (commas are inserted to show separation of words). 00001010101101, 0000101, 011101, 1010101010111.



That's All ; Thank You